

# Equivariant prequantization bundles on the space of connections and characteristic classes

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## Abstract

We show how characteristic classes determine equivariant prequantization bundles over the space of connections on a principal bundle. These bundles are shown to generalize the Chern-Simons line bundles to arbitrary dimensions. Our result applies to arbitrary bundles, and it is studied the action of both the gauge group and the automorphisms group. The action of the elements in the connected component of the identity of the group is computed explicitly, and generalizes known results in the literature. The action of the elements not connected with the identity is shown to be determined by a characteristic class using differential characters and equivariant cohomology.

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## 1 Introduction

In this paper we study the relationship between characteristic classes and equivariant prequantization line bundles over the space of connections. We recall two classical examples of this relation (see Section 2 for the notation).

In the first example, let  $\Sigma$  be a closed (i.e. compact and without boundary) oriented surface,  $P = \Sigma \times SU(2)$  the trivial principal  $SU(2)$ -bundle, and  $p \in I_{\mathbb{Z}}^2(SU(2))$  the polynomial associated to the second Chern class. We denote by  $\mathcal{A}$  the space of connections on  $P$  and by  $\mathcal{G}$  a subgroup of  $\text{Gau}P$ . For simplicity we assume that  $\mathcal{G}$  acts freely on  $\mathcal{A}$  and that  $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$  is a principal  $\mathcal{G}$ -bundle. In [3] Atiyah and Bott show that this polynomial determines a symplectic structure  $\sigma$  on the space of connections  $\mathcal{A}$  which is invariant under the action of  $\mathcal{G}$ . Moreover the curvature map determines a moment map  $\mu$  for  $\sigma$ . By symplectic reduction a symplectic structure  $\underline{\sigma}$  on the moduli space of flat connections  $\mathcal{F}/\mathcal{G}$  is obtained. Furthermore, in [22] it is shown that the action of  $\mathcal{G}$  admits a lift to  $\mathcal{A} \times U(1)$  by  $U(1)$ -bundle automorphisms hence defining a  $\mathcal{G}$ -equivariant  $U(1)$ -bundle over  $\mathcal{A}$  (or what is equivalent, a  $\mathcal{G}$ -equivariant Hermitian line bundle). By taking the quotient, they obtain an Hermitian line bundle  $\mathcal{L} \rightarrow \mathcal{F}/\mathcal{G}$  (which is proved to be isomorphic to the Quillen determinant line bundle) and a natural connection on  $\mathcal{L}$  whose curvature is  $\underline{\sigma}$ . We recall that all this constructions can be done based only on the polynomial  $p$ .

The second example is the classical 3-dimensional Chern-Simons theory. Let  $M$  be a compact 3-dimensional manifold and  $P = M \times SU(2)$  the trivial principal  $SU(2)$ -bundle. If  $M$  is closed then the Chern-Simons action associated to a polynomial  $p \in I_{\mathbb{Z}}^2(SU(2))$  determines a  $\mathcal{G}$ -invariant function  $\mathcal{A} \rightarrow \mathbb{R}/\mathbb{Z}$  and hence a function on the quotient  $\mathcal{A}/\mathcal{G} \rightarrow \mathbb{R}/\mathbb{Z}$ . However, when  $M$  is a manifold with boundary  $\partial M$  the Chern-Simons action is not a function on  $\mathcal{A}/\mathcal{G}$ , but it determines a section of a line bundle  $\mathcal{L}_{\partial M} \rightarrow \mathcal{A}/\mathcal{G}$  called the Chern-Simons line (see e.g. [19]). Again all the constructions are based on a polynomial  $p$ . However, as pointed out in [12], to determine the Chern-Simons action for nontrivial bundles it is also necessary to choose a universal characteristic class  $\Upsilon \in H^4(BG)$  (see also Section 8).

We generalize these two examples to arbitrary bundles, groups and dimensions in the following way. We recall (see [15]) that if  $P \rightarrow M$  is a principal  $G$ -bundle and  $\mathcal{A}$  the space of connections on  $P$ , the principal  $G$ -bundle  $\mathbb{P} = P \times \mathcal{A} \rightarrow M \times \mathcal{A}$  admits a canonical (or tautological) connection  $\mathbb{A}$  which is invariant under the action of the group  $\text{Aut}P$  of automorphisms of  $P$ . If a group  $\mathcal{G}$  acts on  $P \rightarrow M$  by gauge transformations, then for any invariant polynomial  $p \in I_{\mathbb{Z}}^r(G)$  we can consider the  $\mathcal{G}$ -equivariant characteristic forms  $p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^{2r}(M \times \mathcal{A})$  of  $\mathbb{A}$ . If  $c$  is a closed oriented  $d$ -dimensional submanifold of  $M$ , by integrating  $p_{\mathcal{G}}$  over  $c$ , we obtain  $\int_c p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^{2r-d}(\mathcal{A})$  which is closed for the Cartan differential  $D$ . When  $d = 2r - 2$ ,  $\varpi_c = \int_c p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^2(\mathcal{A})$  is a closed equivariant 2-form, i.e.,  $\varpi_c = \sigma_c + \mu_c$  where  $\sigma_c$  is a closed  $\mathcal{G}$ -invariant 2-form and  $\mu_c$  a co-moment map for  $\sigma_c$ . Our main result is the following

**Theorem 1** *Let  $c$  be a closed submanifold of dimension  $2r - 2$  of  $M$ ,  $p \in I_{\mathbb{Z}}^r(G)$ ,  $\Upsilon \in H^{2r}(BG, \mathbb{Z})$  a characteristic class compatible with  $p$  (i.e., they determine the same real characteristic class) and  $A_0$  a background connection on  $P$ . These data determine a lift of the action of  $\mathcal{G}$  on  $\mathcal{A}$  to an action on  $\mathcal{U}_c = \mathcal{A} \times U(1) \rightarrow \mathcal{A}$  by  $U(1)$ -bundle automorphisms, and a  $\mathcal{G}$ -invariant connection form  $\Xi_c$ . Furthermore, the  $\mathcal{G}$ -equivariant curvature of  $\Xi_c$  is  $\varpi_c$  and the first (integer)  $\mathcal{G}$ -*

equivariant Chern class of  $\mathcal{U}_c$  is given by  $c_{1,\mathcal{G}}(\mathcal{U}_c) = \Upsilon_{\mathbb{P}}/c \in H^2(\mathcal{A}_{\mathcal{G}}, \mathbb{Z})$ , where  $\Upsilon_{\mathbb{P}} \in H^{2r}(M \times \mathcal{A}_{\mathcal{G}}, \mathbb{Z})$  is the  $\mathcal{G}$ -equivariant characteristic class of  $\mathbb{P}$  associated to  $\Upsilon$ , and  $/$  denotes cap product.

Due to the equivalence between principal  $U(1)$ -bundles and Hermitian line bundles, we also obtain a  $\mathcal{G}$ -equivariant Hermitian line bundle  $\mathcal{L}_c \rightarrow \mathcal{A}$  with connection  $\nabla^{\Xi_c}$ . Our result also generalizes the Chern-Simons line as we prove the following result.

**Proposition 2** *If  $c = \partial u$  for some  $u \in M$ , then  $S_u(A) = \exp(-2\pi i \int_u Tp(A, A_0))$  determines a  $\mathcal{G}$ -invariant section of  $\mathcal{U}_c \rightarrow \mathcal{A}$ , or what it is equivalent, a  $\mathcal{G}$ -invariant section of unitary norm of  $\mathcal{L}_c \rightarrow \mathcal{A}$ .*

Thus  $p$ ,  $\Upsilon$ ,  $c$  and  $A_0$  determine a  $\mathcal{G}$ -equivariant prequantization bundle for  $(\mathcal{A}, \varpi_c)$ . If we change the background connection  $A_0$  we obtain a different connection, and a different action, but we prove that there exists a canonical  $\mathcal{G}$ -equivariant isomorphisms between them. Therefore we can consider that different background connections  $A_0$  determine different global trivializations of the same prequantization bundle, and hence that it only depends on  $p$ ,  $\Upsilon$  and  $c$ .

It can be shown that for any  $X \in \text{Lie}\mathcal{G}$ , its lift  $X_{\mathcal{U}_c} \in \mathfrak{X}(\mathcal{U}_c)$  does not depend on  $\Upsilon$ . Hence neither does the action of the connected component of the identity  $\mathcal{G}_0$ . An explicit expression for the action of  $\mathcal{G}_0$  is given, which generalizes previous results on [8], [14], [22] and [23]. Nevertheless the action of the elements of  $\mathcal{G}$  which are not connected with the identity depends on  $\Upsilon$ . For certain groups (for example  $U(n)$  or  $SU(n)$ ) there is a bijection  $I_{\mathbb{Z}}^r(G) \simeq H^{2r}(BG, \mathbb{Z})$  and in these cases the action is determined only by  $c$ ,  $p$  and  $A_0$ . But for a general group  $G$  the cohomology  $H^{2r}(BG, \mathbb{Z})$  may contain torsion elements, and  $\Upsilon$  is not determined by  $p$ . In that cases non-equivalent actions can exist with the same  $c$ ,  $p$  and  $A_0$  if  $\mathcal{G}$  is not connected.

We recall that our result is valid for any group action of  $\mathcal{G}$  (not necessarily free). If the action of  $\mathcal{G}$  is free and we have well defined quotient manifolds, we obtain a principal  $U(1)$ -bundle  $\underline{\mathcal{U}}_c = \mathcal{U}_c/\mathcal{G} \rightarrow \mathcal{A}/\mathcal{G}$ , and if  $c = \partial u$  then  $S_u$  determines a section of this bundle. The connection  $\Xi_c$  does not project onto a connection on  $\underline{\mathcal{U}}_c$  as  $\iota_{X_{\mathcal{U}_c}}\Xi_c = -\mu_c(X)$  for  $X \in \text{Lie}\mathcal{G}$ . However, if  $\mathcal{F} \subset \mu_c^{-1}(0)$  is  $\mathcal{G}$ -invariant, the restriction of  $\Xi_c$  to  $\mathcal{F} \times U(1)$  is  $\mathcal{G}$ -basic and projects onto a connection  $\underline{\Xi}_c$  on  $\underline{\mathcal{U}}_c|_{\mathcal{F}/\mathcal{G}} \rightarrow \mathcal{F}/\mathcal{G}$ . Furthermore the curvature of  $\underline{\Xi}_c$  is the form  $\underline{\sigma}_c$  obtained by symplectic reduction of  $(\mathcal{A}, \sigma_c, \mu_c)$ . When  $M$  is a closed surface,  $p$  is the second Chern polynomial and  $c = M$ , then  $\sigma_c$  and  $\mu_c$  coincide with the Atiyah-Bott symplectic structure and moment map (see [15]). Hence our result generalizes that of [22].

We study the dependence on  $c$ . It can be better understood in terms on the Hermitian line bundle  $\mathcal{L}_c \rightarrow \mathcal{A}$ . If  $-c$  denotes the submanifold  $c$  with the opposed orientation, then we have  $\mathcal{L}_{-c} = \mathcal{L}_c^*$ , and if  $c'$  is another closed oriented submanifold then  $\mathcal{L}_{c+c'} \simeq \mathcal{L}_c \otimes \mathcal{L}_{c'}$ . In particular, if  $\partial u = c - c'$  by Proposition 2  $S_u$  determines a section of unitary norm on  $\mathcal{L}_{c-c'} = \mathcal{L}_c \otimes \mathcal{L}_{c'}^* \simeq \text{Hom}(\mathcal{L}_{c'}, \mathcal{L}_c)$  which is an isomorphism.

The symmetry group usually considered in physical theories is the group of gauge transformations. However sometimes it is necessary to consider the lift of the action of the automorphism group  $\text{Aut}P$  to  $\mathcal{U}_c$  (see for example [1, 2] and references therein). We show that Theorem 1 is also valid when  $\mathcal{G}$  acts on  $P$  by automorphisms preserving the orientation of  $M$  in the following cases:

- $M$  is a closed oriented manifold of dimension  $d = 2r - 2$  and  $c = M$ .
- $M$  is a compact oriented manifold of dimension  $d = 2r - 1$  with boundary  $\partial M$  and  $c = \partial M$ . In this case Proposition 2 is also valid.

In this paper we study only the space of connections. However, we formulate our results in such a way that can be applied also to other bundles. For example, in [18] our results are applied to the space of Riemannian metrics and the action of diffeomorphisms. In particular, for a surface, the first Pontryagin class is shown to determine a canonical holomorphic prequantization bundle for the Teichmüller space endowed with the Weil-Petersson symplectic form.

Let us explain the way in which Theorem 1 is obtained. For simplicity we assume that  $\mathcal{G}$  acts freely on  $\mathcal{A}$ . In [4] Chern-Weil theory is applied to the principal  $G$ -bundle  $(P \times \mathcal{A})/\mathcal{G} \rightarrow M \times \mathcal{A}/\mathcal{G}$ . A polynomial  $p \in I_{\mathbb{Z}}^r(G)$  determines a cohomology class  $c_p \in H^{2r}(M \times \mathcal{A}/\mathcal{G})$ , and by integrating this class on a closed  $d$ -dimensional submanifold  $c$  of  $M$ , a cohomology class  $\int_c c_p \in H^{2r-d}(\mathcal{A}/\mathcal{G})$  is obtained. Moreover, as  $p \in I_{\mathbb{Z}}^r(G)$ , we can also apply the Chern-Simons construction to this bundle. We use the Cheeger-Simons approach of [11] based on differential characters. If  $\mathfrak{A}$  is a connection on the principal  $\mathcal{G}$ -bundle  $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$ , it determines a connection on  $(P \times \mathcal{A})/\mathcal{G} \rightarrow M \times \mathcal{A}/\mathcal{G}$  (see below for details) and hence a differential form  $p_{\mathfrak{A}} \in \Omega^{2r}(M \times \mathcal{A}/\mathcal{G})$  whose cohomology class is  $c_p$ . As  $p \in I_{\mathbb{Z}}^r(G)$ , there exists a differential character (the Chern-Simons differential character)  $\chi_{\mathfrak{A}} \in \hat{H}^{2r}(M \times \mathcal{A}/\mathcal{G})$  whose curvature is  $p_{\mathfrak{A}}$ . As pointed out on [11] a Chern-Simons character is determined by a universal characteristic class  $\Upsilon \in \hat{H}^{2r}(BG)$  compatible with  $p$ . By integration over  $c$  we obtain a differential character  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^{2r-d}(\mathcal{A}/\mathcal{G})$  with curvature  $\text{curv}(\int_c \chi_{\mathfrak{A}}) = \int_c p_{\mathfrak{A}} \in \Omega^{2r}(\mathcal{A}/\mathcal{G})$ . We call the characters  $\int_c \chi_{\mathfrak{A}}$  the integrated Chern-Simons characters.

In this paper we study the geometric interpretation of the integrated Chern-Simons characters of order 1 and 2. The geometrical interpretation of higher order differential characters is not so simple, and we postpone it for future research.

As it is well known, a first order differential character can be interpreted as a function  $\mathcal{A}/\mathcal{G} \rightarrow \mathbb{R}/\mathbb{Z}$ . In Section 8 we identify this function with the Dijkgraaf-Witten action for Chern-Simons theory defined in [12].

Our main interest is on second order characters as they lead us to Theorem 1. When  $d = 2r - 2$  we have  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^2(\mathcal{A}/\mathcal{G})$  and, by general results on differential cohomology, it can be represented as the holonomy of a connection on a Hermitian line bundle over  $\mathcal{A}/\mathcal{G}$ . By fixing a background connection  $A_0 \in \mathcal{A}$ , we define an explicit principal  $U(1)$ -bundle  $\mathcal{U}_c \rightarrow \mathcal{A}/\mathcal{G}$  with connection  $\underline{\mathcal{Q}}_c$  and the holonomy of  $\underline{\mathcal{Q}}_c$  is shown to be  $\int_c \chi_{\mathfrak{A}}$ . In more detail, we show that  $A_0$  and  $\int_c \chi_{\mathfrak{A}}$  determine uniquely a lift of the action of  $\mathcal{G}$  on  $\mathcal{A}$  to  $\mathcal{A} \times U(1)$  by  $U(1)$ -

bundle automorphism and a  $\mathcal{G}$ -projectable connection  $\Theta_c$ , and  $\underline{\mathcal{U}}_c$  is defined by setting  $\underline{\mathcal{U}}_c = (\mathcal{A} \times U(1))/\mathcal{G}$ . A similar construction is given in [20] for families of connections, with the assumption that the bundles are trivial and working with local trivializations. Our result applies also for non trivial bundles, and we use a global construction.

The integrated Chern-Simons character  $\int_c \chi_{\mathfrak{A}}$ , the connection  $\Theta_c$  and the action of  $\mathcal{G}$  on  $\mathcal{A} \times U(1)$  are defined in terms of a connection  $\mathfrak{A}$  on the principal  $\mathcal{G}$ -bundle  $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$ . However we prove in Section 6 that the action of  $\mathcal{G}$  on  $\mathcal{A} \times U(1)$  does not depend on the connection  $\mathfrak{A}$  chosen. By using equivariant cohomology we give a version of the preceding results without any reference to the connection  $\mathfrak{A}$ . Finally we show in Section 6.2 that the result is also valid for non-free actions, obtaining in this way Theorem 1.

## 2 Notations and conventions

In this paper we consider two Lie groups. The group  $G$  is the structure group of a principal bundle  $P \rightarrow M$  and it is supposed to be finite dimensional and with a finite number of connected components (in order to apply Chern-Simons construction). The second group  $\mathcal{G}$  is a symmetry group (usually infinite dimensional), and  $\mathcal{G}_0$  denotes the connected component of the identity on  $\mathcal{G}$ .

We denote by  $I_{\mathbb{Z}}^r(G)$  the set of  $G$ -invariant polynomials on its Lie algebra  $\mathfrak{g}$  whose characteristic classes have integral periods. A polynomial  $p \in I_{\mathbb{Z}}^r(G)$  and a characteristic class  $\Upsilon \in H^{2r}(BG, \mathbb{Z})$  are said to be compatible if they determine the same real characteristic class. We denote by  $\mathcal{I}_{\mathbb{Z}}^r = \{(p, \Upsilon) \in I_{\mathbb{Z}}^r(G) \times H^{2r}(BG, \mathbb{Z}) : p, \Upsilon \text{ are compatible}\}$ , and by  $\Upsilon_P$  the characteristic class of  $P \rightarrow M$  associated to  $\Upsilon$ .

The Maurer Cartan form of  $U(1)$  is denoted by  $\theta = u^{-1}du$ , and  $\partial_{\theta} \in \mathfrak{X}(U(1))$  is the vector field such that  $\theta(\partial_{\theta}) = 1$ . If  $\pi: \mathcal{U} \rightarrow N$  is a principal  $U(1)$  bundle and  $\Xi \in \Omega^1(\mathcal{U}, i\mathbb{R})$  is a connection then the curvature form  $\text{curv}(\Xi) \in \Omega^2(N)$  is defined by the property  $\pi^*(\text{curv}(\Xi)) = \frac{i}{2\pi} d\Xi$ . The log-holonomy  $\log \text{hol}_{\Xi}(\gamma) \in \mathbb{R}/\mathbb{Z}$  of  $\Xi$  on a closed curve  $\gamma: I \rightarrow N$  with  $\gamma(0) = \gamma(1)$  is determined by the relation  $\bar{\gamma}(1) = \bar{\gamma}(0) \cdot \exp(2\pi i \log \text{hol}_{\Xi}(\gamma))$ , where  $\bar{\gamma}: I \rightarrow \mathcal{U}$  is the  $\Xi$ -horizontal lift of  $\gamma$ . The (real) first Chern class of  $\mathcal{U}$  is the cohomology class of  $\text{curv}(\Xi)$ . We denote by  $\tilde{x}$  the class of  $x \in \mathbb{R}$  on  $\mathbb{R}/\mathbb{Z}$ .

## 3 Equivariant deRham cohomology in the Cartan model

We recall the definition of equivariant cohomology in the Cartan model (*e.g.* see [7, 21]). Suppose that we have a left action of a connected Lie group  $\mathcal{G}$  on a manifold  $N$ . The map  $X \mapsto X_N(x) = \frac{d}{dt}|_{t=0} (\exp(-tX))(x)$  induces a Lie algebra homomorphism  $\text{Lie } \mathcal{G} \rightarrow \mathfrak{X}(N)$ . The space of  $\mathcal{G}$ -equivariant differential forms is the space of  $\mathcal{G}$ -invariant polynomials on  $\text{Lie } \mathcal{G}$  with values in  $\Omega^{\bullet}(N)$ ,  $\Omega_{\mathcal{G}}(N) = (\mathbf{S}^{\bullet}(\text{Lie } \mathcal{G}^*) \otimes \Omega^{\bullet}(N))^{\mathcal{G}} = \mathcal{P}^{\bullet}(\text{Lie } \mathcal{G}, \Omega^{\bullet}(N))^{\mathcal{G}}$  ( $\mathcal{G}$  acts on  $\text{Lie } \mathcal{G}$

by the adjoint representation). The graduation on  $\Omega_{\mathcal{G}}(N)$  is defined by setting  $\deg(\alpha) = 2k + r$  if  $\alpha \in \mathcal{P}^k(\text{Lie } \mathcal{G}, \Omega^r(N))$ . Let  $D: \Omega_{\mathcal{G}}^q(N) \rightarrow \Omega_{\mathcal{G}}^{q+1}(N)$  be the Cartan differential,  $(D\alpha)(X) = d(\alpha(X)) - \iota_{X_N}\alpha(X)$ , for  $X \in \text{Lie } \mathcal{G}$ . On  $\Omega_{\mathcal{G}}^{\bullet}(N)$  we have  $D^2 = 0$ , and the  $\mathcal{G}$ -equivariant cohomology (in the Cartan model) of  $N$  is defined as the cohomology of this complex.

Next, let us recall the relationship between equivariant cohomology and the cohomology of the quotient space in the case in which the action of  $\mathcal{G}$  on  $N$  is free and  $\pi: N \rightarrow N/\mathcal{G}$  is a principal  $\mathcal{G}$ -bundle (this happens for example if  $\mathcal{G}$  is compact or if the action of  $\mathcal{G}$  on  $N$  is proper). We recall that if the action of  $\mathcal{G}$  on  $N$  is on the left, it defines also a right action by  $R_{\phi}x = \phi^{-1}x$  for  $\phi \in \mathcal{G}$  and  $x \in N$ . If  $A$  is a connection on  $N \rightarrow N/\mathcal{G}$ , the following map is a generalization of the Chern-Weil homomorphism  $\text{Cw}_A: \Omega_{\mathcal{G}}^{\bullet}(N) \rightarrow \Omega^{\bullet}(N/\mathcal{G}) \simeq (\Omega^{\bullet}(N))_{A\text{-basic}}$ , determined by  $\pi^*(\text{Cw}_A(\alpha)) = (\alpha(F_A))_{\text{hor } A}$ , where  $\beta_{\text{hor } A}$  denotes the horizontalization of  $\beta \in \Omega^*(N)$  with respect to the connection  $A$  and  $F_A$  the curvature of  $A$ . We have  $\text{Cw}_A(D\alpha) = d(\text{Cw}_A(\alpha))$ , and the induced map in cohomology  $\text{Cw}_A: H_{\mathcal{G}}^{\bullet}(N) \rightarrow H^{\bullet}(N/\mathcal{G})$  is independent of the connection  $A$  chosen and is denoted by  $\text{Cw}_N$ . We study the map  $\text{Cw}_A$  in the two simplest cases:

### 3.1 Equivariant 1-forms

A  $\mathcal{G}$ -equivariant 1-form is simply a  $\mathcal{G}$ -invariant 1-form  $\alpha$ , and  $\pi^*\text{Cw}_A(\alpha) = \alpha_{\text{hor } A} = \alpha - \alpha(A)$ , where  $\alpha(A(Y)) = \alpha_y(A(Y)_N)$  for  $Y \in T_yN$ . The form  $\alpha$  is  $D$  closed if  $d\alpha = 0$  and  $\iota_{X_N}\alpha = 0$  for every  $X \in \text{Lie } \mathcal{G}$ . In this case  $\alpha$  projects onto a closed 1-form  $\underline{\alpha} \in \Omega^1(N/\mathcal{G})$ . We conclude that if  $\alpha$  is  $D$ -closed, then  $\text{Cw}_A(\alpha) = \underline{\alpha}$  does not depend on the connection  $A$  chosen. We also see that  $\text{Cw}_N: H_{\mathcal{G}}^1(N) \rightarrow H^1(N/\mathcal{G})$  induces an isomorphism in cohomology.

### 3.2 Equivariant 2-forms

A  $\mathcal{G}$ -equivariant 2-form  $\varpi$  is given by  $\varpi(X) = \sigma + \mu(X)$  where  $\sigma$  is a  $\mathcal{G}$ -invariant 2-form and  $\mu: \text{Lie } \mathcal{G} \rightarrow \Omega^0(N)$  a linear  $\mathcal{G}$ -equivariant map. The form  $\varpi$  is  $D$ -closed if  $d\sigma = 0$  and  $\iota_{X_N}\sigma = \mu(X)$  for every  $X$ . Hence  $\mu$  is a co-moment map for  $\sigma$ . We use below the following result

**Lemma 3** *If  $\varpi = \sigma + \mu$  is  $D$ -closed then we have  $\pi^*\text{Cw}_A(\varpi) = \varpi + D(\mu(A)) = \sigma + d(\mu(A))$ . In particular, for two connections  $A, A'$  we have  $\pi^*\text{Cw}_A(\alpha) - \pi^*\text{Cw}_{A'}(\alpha) = d(\mu(A - A'))$ .*

### 3.3 Equivariant characteristic classes in the Cartan model

We recall the definition of equivariant characteristic classes (see [6, 9] for details). Let  $\mathcal{G}$  be a group that acts (on the left) on a principal  $G$ -bundle  $\pi: P \rightarrow M$  and let  $A$  be a connection on  $P$  invariant under the action of  $\mathcal{G}$ . It can be proved (see [6, 9]) that for every  $X \in \text{Lie } \mathcal{G}$  the  $\mathfrak{g}$ -valued function  $A(X_P)$  is of adjoint type and defines a section of the adjoint bundle  $v_A(X) \in \Omega^0(N, \text{ad } P)$ . For every  $p \in I^r(G)$  the  $\mathcal{G}$ -equivariant characteristic form  $p_{\mathcal{G}, A} \in \Omega_{\mathcal{G}}^{2k}(N)$  associated to  $p$

and  $A$ , is defined by  $p_{\mathcal{G},A}(X) = p(F_A - v_A(X))$  for every  $X \in \text{Lie}\mathcal{G}$ . If  $\mathcal{G}$  acts freely on  $P$  and  $N$ , and we have a quotient bundle  $\pi_{\mathcal{G}}: P/\mathcal{G} \rightarrow N/\mathcal{G}$ , then we have the following (see [10] for details and also [13, Section 5.2.3]):

**Proposition 4** *If  $A_1$  is a  $\mathcal{G}$ -invariant connection on  $P \rightarrow N$  and  $A_2$  a connection on  $\pi_{\mathcal{G}}: N \rightarrow N/\mathcal{G}$  then  $A = A_1 - v_{A_1}(\pi^*A_2)$  projects onto a connection  $\underline{A}$  on  $P/\mathcal{G} \rightarrow N/\mathcal{G}$ , and we have  $\pi_{\mathcal{G}}^*F_{\underline{A}} = (F_1)_{\text{hor}A_2} - v_{A_1}(\pi_{\mathcal{G}}^*F_2)$ . Hence  $\text{Cw}_{A_2}(p_{\mathcal{G},A_1}) = p(F_{\underline{A}})$ .*

A  $\mathcal{G}$ -equivariant  $U(1)$ -bundle is a principal  $U(1)$ -bundle  $\mathcal{U} \rightarrow N$  in which  $\mathcal{G}$  acts by  $U(1)$ -bundle automorphisms. If  $\Xi \in \Omega^1(\mathcal{U}, i\mathbb{R})$  is a  $\mathcal{G}$ -invariant connection then  $\frac{i}{2\pi}D(\Xi)$  projects onto a closed  $\mathcal{G}$ -equivariant 2-form  $\text{curv}_{\mathcal{G}}(\Xi) \in \Omega_{\mathcal{G}}^2(N)$  called the  $\mathcal{G}$ -equivariant curvature of  $\Xi$ . If  $X \in \text{Lie}\mathcal{G}$  then  $\text{curv}_{\mathcal{G}}(\Xi)(X) = \text{curv}(\Xi) - \frac{i}{2\pi}\iota_{X_{\mathcal{U}}}\Xi$ . If  $\varpi \in \Omega_{\mathcal{G}}^2(N)$ , a  $\mathcal{G}$ -equivariant pre-quantization bundle for  $\varpi$  is a principal  $U(1)$ -bundle  $\mathcal{U} \rightarrow N$  with connection  $\Xi$  such that  $\text{curv}_{\mathcal{G}}(\Xi) = \varpi$ .

### 3.4 Equivariant characteristic classes in the Borel Model

The Cartan model of equivariant cohomology allows us to study equivariant de Rham cohomology. However, we also need to consider integer cohomology, and we need another model, the Borel model (see for example [3]). Let  $\mathcal{G}$  be a group acting on a manifold  $N$  and let  $E\mathcal{G} \rightarrow B\mathcal{G}$  be a universal principal  $\mathcal{G}$ -bundle, i.e.  $E\mathcal{G}$  is a contractible space. As it is well known, every principal  $\mathcal{G}$ -bundle is a pull-back of the bundle  $E\mathcal{G} \rightarrow B\mathcal{G}$  by a map  $f: N \rightarrow B\mathcal{G}$ . The homotopy quotient of  $N$  is defined as the space  $N_{\mathcal{G}} = N \times_{\mathcal{G}} E\mathcal{G}$ , and the cohomology  $H^{\bullet}(N_{\mathcal{G}}, \mathbb{Z})$  is called the  $\mathcal{G}$ -equivariant cohomology of  $N$  in the Borel model. When  $N \rightarrow N/\mathcal{G}$  is a principal  $\mathcal{G}$ -bundle the projection  $N_{\mathcal{G}} \rightarrow N/\mathcal{G}$  induces a map  $H^{\bullet}(N/\mathcal{G}, \mathbb{Z}) \rightarrow H^{\bullet}(N_{\mathcal{G}}, \mathbb{Z})$  which is an isomorphism if  $\mathcal{G}$  is compact. The relationship between the equivariant cohomology in the Cartan and Borel model is given by the map  $\text{Cw}_{N \times E\mathcal{G}} \circ q^*: H_{\mathcal{G}}^k(N) \rightarrow H^k(N_{\mathcal{G}}, \mathbb{R})$ , where  $q: N \times E\mathcal{G} \rightarrow N$  denotes the projection. If  $\mathcal{G}$  is a compact group this map is an isomorphism. If  $P \rightarrow N$  is a principal  $\mathcal{G}$ -bundle in which  $\mathcal{G}$  acts by automorphism, and  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ , then the  $\mathcal{G}$ -equivariant characteristic class in the Borel model  $\Upsilon_{N_{\mathcal{G}}} \in H^{2r}(N_{\mathcal{G}}, \mathbb{Z})$  is the characteristic class of the bundle  $P_{\mathcal{G}} \rightarrow N_{\mathcal{G}}$ . We have  $\text{Cw}_{N \times E\mathcal{G}} \circ q^*(p_{\mathcal{G}}) = r(\Upsilon_{N_{\mathcal{G}}})$  where  $r$  denotes the map  $H^{2r}(N_{\mathcal{G}}, \mathbb{Z}) \rightarrow H^{2r}(N_{\mathcal{G}}, \mathbb{R})$ .

## 4 Cheeger-Simons differential characters

We recall the definition of differential characters (see [11] and [5] for details). We denote by  $C_k(N)$  and  $Z_k(N)$  the smooth chains and cycles on  $N$ . A Cheeger-Simons differential character of order  $k$  is a homomorphism  $\chi: Z_{k-1}(N) \rightarrow \mathbb{R}/\mathbb{Z}$  such that there exist  $\alpha \in \Omega^k(N)$  with satisfies  $\chi(\partial u) = \int_u \alpha$  for every  $u \in C_k(N)$ . We say that  $\chi$  is a differential character with curvature  $\text{curv}(\chi) = \alpha$  and it can be proved that  $d\text{curv}(\chi) = 0$ . We denote the space of differential characters of order  $k$  on  $N$  by  $\hat{H}^k(N)$ . We have a map  $\text{char}: \hat{H}^k(N) \rightarrow H^k(N, \mathbb{Z})$ , and the class  $\text{char}(\chi)$  is called the characteristic class of  $\chi$ . The maps  $\text{char}$  and  $\text{curv}$

are compatible in the sense that we have  $r(\text{char}(\chi)) = [\text{curv}(\chi)] \in H^k(N, \mathbb{R})$ . If  $f: N' \rightarrow N$  is a smooth map, it induces a map  $f^*: \hat{H}^k(N) \rightarrow \hat{H}^k(N')$ . Given  $\beta \in \widetilde{\Omega^{k-1}(N)}$  we define a differential character  $\varsigma(\beta) \in \hat{H}^k(N)$  by setting  $\varsigma(\beta)(s) = \widetilde{\int_s \beta}$  for  $s \in \widetilde{Z_{k-1}(M)}$ . We have  $\text{curv}(\varsigma(\beta)) = d\beta$ , and  $\text{char}(\varsigma(\beta)) = 0$ . Note that  $\varsigma(d\alpha)(s) = \widetilde{\int_s d\alpha} = \widetilde{\int_{\partial s} \alpha} = 0$  as  $\partial s = 0$ .

The differential characters of low degree have the following geometrical interpretation, that we use below

#### 4.1 Differential characters of order 1

A differential character of order 1 is a homomorphism  $\chi: Z_0(M) \rightarrow \mathbb{R}/\mathbb{Z}$  such that there exist  $\lambda \in \Omega^1(M)$  with  $\chi(\{x\} - \{y\}) = \int_\gamma \lambda$  for any  $\gamma \subset M$ ,  $\partial\gamma = x - y$ . We define  $\varphi_\chi: M \rightarrow \mathbb{R}/\mathbb{Z}$  by  $\varphi_\chi(x) = \chi(\{x\})$ . It can be seen that  $\varphi$  is differentiable (see [5]). Conversely, given any map  $\varphi: M \rightarrow \mathbb{R}/\mathbb{Z}$ , we can define  $\chi_\varphi: Z_0(M) \rightarrow \mathbb{R}/\mathbb{Z}$  by setting  $\chi_\varphi(\{x\}) = \varphi(x)$ , and  $\lambda = \varphi^*(dh)$  where  $h$  is a coordinate on  $\mathbb{R}$ . Hence, a differential character of order 1 is simply a map  $\varphi: M \rightarrow \mathbb{R}/\mathbb{Z}$ .

#### 4.2 Differential characters of order 2

First we recall that if  $\mathcal{U} \rightarrow M$  a principal  $U(1)$ -bundle with connection  $\Theta$ , and curvature  $\omega \in \Omega^2(M)$ , the log-holonomy of  $\Theta$   $\log \text{hol}_\Theta: Z_1(M) \rightarrow \mathbb{R}/\mathbb{Z}$  is a differential character with curvature  $\text{curv}(\log \text{hol}_\Theta) = \omega$  and  $\text{char}(\log \text{hol}_\Theta) = c_1(\mathcal{U})$ . Conversely, by a classical result in differential cohomology, every second order differential character can be represented as the holonomy of a connection  $\Theta$  on a principal  $U(1)$  bundle  $\mathcal{U} \rightarrow M$ . The bundle  $\mathcal{U}$  and the connection  $\Theta$  are determined by  $\chi$  only modulo isomorphisms. In the next Proposition we show that in the following restrictive equivariant case it is possible to give a concrete bundle and connection

**Theorem 5** *Let  $N$  be a connected and simply connected manifold in which  $\mathcal{G}$  acts in such a way that  $\pi: N \rightarrow N/\mathcal{G}$  is a principal  $\mathcal{G}$ -bundle. Let  $\chi \in \hat{H}^2(N/\mathcal{G})$  be a second order differential character on  $N/\mathcal{G}$  with curvature  $\omega$ , and assume that there exist  $\lambda \in \Omega^1(N)$  such that  $\pi^*\omega = d\lambda$ . Then there exists a unique lift of the action of  $\mathcal{G}$  to  $N \times U(1)$  by  $U(1)$ -bundle automorphism such that  $\Theta = \theta - 2\pi i\lambda \in \Omega^1(N \times U(1), i\mathbb{R})$  is projectable onto a connection  $\underline{\Theta}$  on  $\mathcal{U} = (N \times U(1))/\mathcal{G} \rightarrow N/\mathcal{G}$  and  $\chi = \log \text{hol}_{\underline{\Theta}}$ . The action of  $\phi \in \mathcal{G}$  on  $N \times U(1)$  is given by  $\Phi: N \times U(1) \rightarrow N \times U(1)$ ,  $\Phi(x, u) = (\phi x, \exp(2\pi i\alpha_\phi(x)) \cdot u)$ , where  $\alpha_\phi: N \rightarrow \mathbb{R}/\mathbb{Z}$  is defined by  $\alpha_\phi(x) = \widetilde{\int_\gamma \lambda} - \chi(\pi \circ \gamma)$ , and  $\gamma$  is any curve on  $N$  joining  $x$  and  $\phi x$ .*

The proof of Theorem 5 is a consequence of the following lemmas. If  $\gamma$  is a curve on  $N$ , we denote by  $\underline{\gamma} = \pi \circ \gamma$  the projected curve on  $N/\mathcal{G}$ .

**Lemma 6** *Let  $\gamma$  and  $\gamma'$  be two curves on  $N$  joining  $x$  and  $\phi x$ . Then  $\widetilde{\int_\gamma \lambda} - \chi(\underline{\gamma}) = \widetilde{\int_{\gamma'} \lambda} - \chi(\underline{\gamma'})$ .*



**Proof.** As  $N$  is simply connected we have  $\gamma - \gamma' = \partial D$  on  $N$ . If  $\underline{D} = \pi \circ D$ , then  $\underline{\gamma} - \underline{\gamma}' = \partial \underline{D}$ , and hence  $\chi(\underline{\gamma}) - \chi(\underline{\gamma}') = \chi(\underline{\gamma} - \underline{\gamma}') = \int_{\underline{D}} \omega = \int_D d\lambda = \int_{\gamma} \lambda - \int_{\gamma'} \lambda \bmod \mathbb{Z}$ . ■

For every  $\phi \in \mathcal{G}$  we define  $\alpha_\phi: N \rightarrow \mathbb{R}/\mathbb{Z}$  by  $\alpha_\phi(x) = \int_{\gamma} \lambda - \chi(\underline{\gamma})$ , where  $\gamma$  is a curve on  $N$  joining  $x$  and  $\phi x$  (it is well defined by the preceding Lemma).

**Lemma 7** a) We have  $\alpha_\phi(x') = \alpha_\phi(x) + \int_{\gamma_{xx'}} (\phi^* \lambda - \lambda)$  for any curve  $\gamma_{xx'}$  joining  $x$  and  $x'$ .

b) As a consequence of a) we have  $d\alpha_\phi = \phi^* \lambda - \lambda$ .

c) If  $\phi_t$  is a 1-parameter group on  $\mathcal{G}$  with  $\dot{\phi} = X$ , then  $\left. \frac{d\alpha_{\phi_t}(x)}{dt} \right|_{t=0} = \lambda(X_N)(x)$ .

**Proof.** a) If  $\gamma$  is a curve joining  $x$  and  $\phi x$ ,  $\gamma_{x'x}$  a curve joining  $x'$  and  $x$ , and  $\gamma_{xx'}$  the inverse curve. Then  $\gamma' = \gamma_{x'x} * \gamma * \phi \gamma_{xx'}$  is a curve joining  $x'$  and  $\phi x'$ . Clearly  $\underline{\gamma}' = \underline{\gamma}$  on  $Z_1(N/\mathcal{G})$ , and hence  $\chi(\underline{\gamma}) = \chi(\underline{\gamma}')$ . We have

$$\begin{aligned} \alpha_\phi(x') &= \int_{\gamma'} \lambda - \chi(\underline{\gamma}') = \int_{\gamma_{x'x}} \lambda + \int_{\gamma} \lambda + \int_{\phi \gamma_{xx'}} \lambda - \chi(\underline{\gamma}) \\ &= \alpha_\phi(x) + \int_{\gamma_{xx'}} \phi^* \lambda - \int_{\gamma_{xx'}} \lambda = \alpha_\phi(x) + \int_{\gamma_{xx'}} (\phi^* \lambda - \lambda) \end{aligned}$$

b) If  $\gamma$  is a curve with  $\gamma(0) = x$  and  $\gamma'(0) = X \in T_x N$ , we have  $\alpha_\phi(\gamma(s)) = \alpha_\phi(x) + \int_0^s (\phi^* \lambda - \lambda)_{\gamma(s)}(\gamma'(s)) ds$ , and hence  $d\alpha_\phi(x)(X) = \left. \frac{d\alpha_\phi(x_s)}{ds} \right|_{s=0} = (\phi^* \lambda - \lambda)_x(X)$ .

c) Let  $X \in \text{Lie } \mathcal{G}$  and  $\phi_t$  a 1-parameter group on  $\mathcal{G}$  with  $\phi_0 = 1_{\mathcal{G}}$  and  $\dot{\phi}_0 = X$ , and we define  $\gamma(t) = \phi_t x$ . We have  $\underline{\gamma} = 0$  on  $Z_1(N/\mathcal{G})$  and  $\dot{\gamma}(0) = X_N(x)$ . Then  $\alpha_{\phi_t}(x) = \int_{\gamma} \lambda - \chi(\underline{\gamma}) = \int_{\gamma} \lambda = \int_0^t \lambda_{\phi_s x}(\dot{\gamma}(s)) ds$  and  $\left. \frac{d\alpha_{\phi_t}(x)}{dt} \right|_{t=0} = \lambda(X_N(x))$ . ■

The action  $\alpha$  satisfies the following cocycle condition

**Lemma 8** We have  $\alpha_{\phi_2 \phi_1}(x) = \alpha_{\phi_1}(x) + \alpha_{\phi_2}(\phi_1 x)$  for any  $x \in N$ ,  $\phi_1, \phi_2 \in \mathcal{G}$ .

**Proof.** Let  $\gamma_1$  be a curve joining  $x$  and  $\phi_1 x$  and  $\gamma_2$  be a curve joining  $\phi_1 x$  and  $\phi_2 \phi_1 x$ . Then  $\gamma' = \gamma_2 * \gamma_1$  is a curve joining  $x$  and  $\phi_2 \phi_1 x$ . We have  $\underline{\gamma}' = \underline{\gamma}_2 + \underline{\gamma}_1$  on  $Z_1(N/\mathcal{G})$ . Hence

$$\alpha_{\phi_2 \phi_1}(x) = \int_{\gamma'} \lambda - \chi(\underline{\gamma}') = \int_{\gamma_2} \lambda + \int_{\gamma_1} \lambda - \chi(\underline{\gamma}_1) - \chi(\underline{\gamma}_2) = \alpha_{\phi_1}(x) + \alpha_{\phi_2}(\phi_1 x).$$

■

We define the action of  $\mathcal{G}$  on  $N \times U(1)$  by  $\Phi(x, u) = (\phi x, \exp(2\pi i \alpha_\phi(x)) \cdot u)$ . It defines a group action as we have

$$\begin{aligned} \Phi_2(\Phi_1(x, u)) &= \Phi_2((\phi_1 x, \exp(2\pi i \alpha_{\phi_1}(x)) \cdot u)) \\ &= (\phi_2 \phi_1 x, \exp(2\pi i (\alpha_{\phi_1}(x) + \alpha_{\phi_2}(\phi_1 x))) \cdot u) \\ &= (\phi_2 \phi_1 x, \exp(2\pi i \alpha_{\phi_2 \phi_1}(x)) \cdot u) = (\Phi_2 \Phi_1)(x, u) \end{aligned}$$

We also define  $\Theta = \theta - 2\pi i\lambda \in \Omega^1(N \times U(1), i\mathbb{R})$ ,  $\mathcal{U} = (N \times U(1))/\mathcal{G}$  and we denote by  $\bar{\pi}: N \times U(1) \rightarrow \mathcal{U}$  the projection. For every  $\phi \in \mathcal{G}$  we have

$$\Phi^*\Theta = \Phi^*\theta - 2\pi i\phi^*\lambda = (\theta + 2\pi i d\alpha_\phi) - 2\pi i\phi^*\lambda = (\theta + 2\pi i(\phi^*\lambda - \lambda)) - 2\pi i\phi^*\lambda = \Theta.$$

Moreover, for every  $X \in \text{Lie}\mathcal{G}$ , if  $\phi_t$  is a curve on  $\mathcal{G}$  with  $X = \dot{\phi}$  then we have  $X_{N \times U(1)} = X_N + 2\pi i \frac{d\alpha_{\phi_t}(x)}{dt} \Big|_{t=0} \partial_\theta = X_N + 2\pi i\lambda(X_N)\partial_\theta$ , and hence  $\Theta(X_{N \times U(1)}) = 0$ . We conclude that  $\Theta$  is a  $\mathcal{G}$ -basic form, i.e. there exists  $\underline{\Theta} \in \Omega^1(N/\mathcal{G}, i\mathbb{R})$  such that  $\bar{\pi}^*\underline{\Theta} = \Theta$ . Clearly  $\Theta$  is a connection form.

**Lemma 9** *We have  $\log \text{hol}_{\underline{\Theta}} = \chi$ .*

**Proof.** Given a loop  $\underline{\gamma}$  on  $N/\mathcal{G}$  with  $\underline{\gamma}(0) = \underline{\gamma}(1) = [x] \in N/\mathcal{G}$ , we can find a curve  $\gamma$  on  $N$  with  $\gamma(0) = x$  such that  $\pi \circ \gamma = \underline{\gamma}$ . We have  $\gamma(1) = \phi x$  for some  $\phi \in \mathcal{G}$ . The  $\Theta$ -horizontal lift of  $\gamma$  to  $N \times U(1)$  starting at the point  $(x, 0)$  is given by  $\bar{\gamma}(s) = (\gamma(s), \exp(2\pi i \int_0^s \lambda_{\gamma(t)}(\dot{\gamma}_k(t)) dt))$ . The curve  $\bar{\pi} \circ \bar{\gamma}$  is a  $\underline{\Theta}$ -horizontal lift to  $\mathcal{U}$  of the loop  $\underline{\gamma} = \pi \circ \gamma_k$ . In particular we have  $\bar{\pi} \circ \bar{\gamma}(1) = (\phi x, \exp(2\pi i \int_\gamma \lambda)) \sim_{\mathcal{G}} (x, \exp(2\pi i(\int_\gamma \lambda - \alpha_\phi(x))))$ . Hence  $\log \text{hol}_{\underline{\Theta}}(\underline{\gamma}) = \int_\gamma \lambda - \alpha_\phi(x) = \chi(\underline{\gamma})$ . ■

The proof of the preceding Proposition also shows that the action we have defined is the unique with satisfies  $\log \text{hol}_{\underline{\Theta}}(\underline{\gamma}) = \chi(\underline{\gamma})$ . This is equivalent to  $\chi(\underline{\gamma}) = \int_\gamma \lambda - \alpha_\phi(x)$ , and hence  $\alpha_\phi(x) = \int_\gamma \lambda - \chi(\underline{\gamma})$ , that is our definition of  $\alpha_\phi(x)$ .

For the elements in  $\mathcal{G}_0$  (the connected component of the identity in  $\mathcal{G}$ ) we have a simpler result:

**Proposition 10** *Let  $\phi \in \mathcal{G}_0$  and  $\varphi \subset \mathcal{G}$  is a curve such that  $\varphi_0 = 1_{\mathcal{G}}$  and  $\varphi_1 = \phi$ . Then  $\alpha_\phi(x) = \int_{\varphi \cdot x} \lambda$ .*

**Proof.** The curve  $\gamma = \varphi \cdot x$  is a curve joining  $x$  and  $\phi x$ , and  $\underline{\gamma} = 0$  on  $Z_1(N/\mathcal{G})$ . Hence  $\alpha_\phi(x) = \int_\gamma \lambda - \chi(\underline{\gamma}) = \int_{\varphi x} \lambda$ . ■

**Remark 11** The preceding Proposition determines the action of  $\mathcal{G}_0$  only in terms of  $\lambda$ , and without any reference to  $\chi$ . Hence the differential character  $\chi$  is necessary only to determine the action of the elements of  $\mathcal{G}$  not connected with the identity. We note that Theorem 5 is a generalization to non connected groups of results in [8], [14] and [23] for the space of connections.

### 4.3 Chern-Simons differential characters

Chern-Simons theory allows to find, in a natural way, a differential character with curvature a characteristic form. Let  $G$  be a Lie group with a finite number of connected components,  $p \in I_{\mathbb{Z}}^r(G)$  and  $q: P \rightarrow N$  a principal  $G$ -bundle over a manifold  $N$ . If  $A$  is a connection on  $q: P \rightarrow N$  with curvature  $F$ , we have  $p(F) \in \Omega^k(N)$ . It can be seen (see [11]) that if  $\Upsilon \in H^{2r}(BG, \mathbb{Z})$  is a universal characteristic class compatible with  $p$ , there exist a differential character  $\chi^A$  such that  $\text{curv}(\chi^A) = p(F)$  and  $\text{char}(\chi) = \Upsilon_P$ . We call  $\chi^A$  the Chern-Simons character of  $p, \Upsilon$  and  $A$ . The Chern-Simons character is characterized

as being the unique natural map  $(P, A) \mapsto \chi^A$  satisfying  $\text{curv}(\chi^A) = p(F)$  and  $\text{char}(\chi^A) = \Upsilon_P$ . We recall that natural means that for any principal  $G$ -bundle  $P' \rightarrow N'$  and any  $G$ -bundle map  $f: P' \rightarrow P$  we have  $\chi^{f^*A} = \underline{f}^*(\chi^A)$ , where  $\underline{f}: N' \rightarrow N$  is the map induced by  $f$ . Furthermore, if  $A'$  is another connection on  $P$ , then we have  $\chi^{A'} = \chi^A + \varsigma(Tp(A', A))$ .

**Remark 12** The original Chern-Simons and Cheeger-Simons constructions are valid for finite dimensional manifolds, but they can be extended to Banach or Fréchet infinite dimensional manifolds, and to more general types of spaces (see for example [5]). Hence they can be applied to the infinite dimensional spaces of connections and metrics.

## 4.4 Fiber integration of differential characters

### 4.4.1 Integration on a product

If  $\alpha \in \Omega^k(C \times S)$  with  $C$  compact and  $\dim C = d$  we define  $\int_C \alpha \in \Omega^{k-d}(S)$  by  $(\int_C \alpha)_s(X_1, \dots, X_{k-d}) = \int_C \iota_{X_{k-d}} \cdots \iota_{X_1} \alpha_s$  for  $s \in S$ ,  $X_1, \dots, X_d \in T_s S$ . If  $k < d$  we define  $\int_C \alpha = 0$ . We have  $\int_S \int_C \alpha = \int_{C \times S} \alpha$ , and also  $\int_C \alpha = \int_C \alpha^{d, k-d}$ , where  $\alpha^{d, k-d}$  is the component relative to the bigraduation associated to the product structure on  $C \times S$ . Furthermore we have Stokes theorem  $d \int_C \alpha = \int_C d\alpha - (-1)^{k-d} \int_{\partial C} \alpha$ . If  $c: C \rightarrow M$  is a map with  $\dim C = d$  we define maps  $\int_c: \Omega^k(M \times N) \rightarrow \Omega^{k-d}(N)$  by  $\int_c \alpha = \int_C (c \times \text{id}_N)^* \alpha$  and we have  $\int_c \alpha = \int_C \alpha^{d, k-d}$ . If a group acts on  $M \times N$  and  $C$  and the map  $c$  is  $\mathcal{G}$ -equivariant this map is easily extended to equivariant differential forms  $\int_c: \Omega_{\mathcal{G}}^k(M \times N) \rightarrow \Omega_{\mathcal{G}}^{k-d}(N)$  and we have  $D \int_c \alpha = \int_C D\alpha - (-1)^{k-d} \int_{\partial C} \alpha$ . If the action of  $\mathcal{G}$  on  $N$  is free and  $\mathfrak{A}$  is a connection on  $N \rightarrow N/\mathcal{G}$  we have  $\int_c \text{Cw}_{\mathfrak{A}}(\alpha) = \text{Cw}_{\mathfrak{A}}(\int_c \alpha)$  for every  $\alpha \in \Omega_{\mathcal{G}}^k(M \times N)$  (for simplicity we use the same letter for the connection  $\mathfrak{A}$  and the connection induced by  $\mathfrak{A}$  on  $M \times N \rightarrow (M \times N)/\mathcal{G}$ ).

The integration map can be extended to differential characters in the following way. If  $\chi \in \hat{H}^n(M \times N)$  is a differential character of order  $n$  on  $M \times N$  and  $c: C \rightarrow M$  is a smooth map with  $C$  closed, we define  $\int_c \chi \in \hat{H}^{n-d}(N)$  by  $(\int_c \chi)(s) = \chi(c \times s)$ , and we have  $\int_c \chi(\partial t) = \chi(c \times \partial t) = \int_{c \times t} \text{curv}(\chi) = \int_t \int_c \text{curv}(\chi)$ . Hence  $\int_c \chi$  is a differential character on  $N$  and its curvature is  $\text{curv}(\int_c \chi) = \int_c \text{curv}(\chi)$ . Moreover, if  $c = \partial u$  for some  $u: U \rightarrow M$  we have  $\int_{\partial u} \chi = \varsigma(\int_u \text{curv}(\chi))$  as  $\int_{\partial u} \chi(s) = \chi(\partial u \times s) = \chi(\partial(u \times s)) = \int_{u \times s} \text{curv}(\chi) = \int_s \int_u \text{curv}(\chi)$ .

### 4.4.2 Fiber integration

The integration of differential characters can be extended to fiber integration on a nontrivial bundle  $\mathcal{N} \rightarrow N$  with fibre  $M$  (e.g. see [5, 20]). In the product case we can integrate over any submanifold of  $M$ , but for nontrivial bundles it only makes sense integration over the fiber  $M$  and over  $\partial M$  if the fiber has boundary. If  $\mathcal{N} \rightarrow N$  is a fiber bundle with compact and oriented fibre  $M$  without boundary

of dimension  $d$ , the fiber integration is a map  $\int_M: \hat{H}^n(\mathcal{N}) \rightarrow \hat{H}^{n-d}(N)$  and satisfies  $\text{curv}(\int_M \chi) = \int_M \text{curv}(\chi)$ , and  $\text{char}(\int_M \chi) = \int_M \text{char}(\chi)$ . If  $M$  has boundary we have a map  $\int_{\partial M}: \hat{H}^n(\mathcal{N}) \rightarrow \hat{H}^{n-d+1}(N)$  and

$$\int_{\partial M} \chi = \varsigma(\int_M \text{curv}(\chi)). \quad (1)$$

## 5 Integrated Chern-Simons characters

### 5.1 Definition of integrated Chern-Simons characters

In the present section we define the integrated Chern-Simons characters for a general connected and simply connected manifold  $N$  that. We show below and in [18], that it includes the case of the spaces of connections and metrics as particular cases. We assume that  $\mathcal{G}$  is a Lie group that acts (on the left) on the following spaces

- a) on a principal  $G$ -bundle  $P \rightarrow M$  by  $G$ -bundle automorphisms,
- b) on a closed oriented manifold  $C$  of dimension  $d$  and we have a  $\mathcal{G}$ -equivariant map  $c: C \rightarrow M$  and  $\mathcal{G}$  preserves the orientation of  $C$ .
- c) on a manifold  $N$  and there exist a  $\mathcal{G}$ -invariant connection  $\mathbb{A}$  on the product bundle  $\mathbb{P} = P \times N \rightarrow M \times N$ .

In this section we also make the following assumption:

- d) the action of  $\mathcal{G}$  on  $N$  is free and  $N \rightarrow N/\mathcal{G}$  is a principal  $\mathcal{G}$ -bundle.

The following definitions are generalizations of the results on [15] for connections and [17] for Riemannian metrics. As the connection  $\mathbb{A}$  is  $\mathcal{G}$ -invariant, for any  $p \in I^r(G)$  we can define the  $\mathcal{G}$ -equivariant characteristic class  $p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^{2r}(M \times N)$ , given by  $p_{\mathcal{G}}(X) = p(\mathbb{F} - v_{\mathbb{A}}(X))$  for  $X \in \text{Lie}\mathcal{G}$ . As  $c: C \rightarrow M$  is  $\mathcal{G}$ -invariant we can integrate over  $C$  to obtain  $\int_c p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^{2r-d}(N)$ .

If  $\mathfrak{A}$  is a connection on  $N \rightarrow N/\mathcal{G}$ , it induces a connection on  $(M \times N) \rightarrow (M \times N)/\mathcal{G}$ , that we denote by the same letter. We have the quotient principal  $G$ -bundle  $\mathbb{P}/\mathcal{G} \rightarrow (M \times N)/\mathcal{G}$  and by Proposition 4 the connections  $\mathbb{A}$  and  $\mathfrak{A}$  determine a connection  $\Lambda_{\mathfrak{A}}$  on this bundle with  $p$ -characteristic form  $\underline{p}_{\mathfrak{A}} = \text{Cw}_{\mathfrak{A}}(p_{\mathcal{G}}) \in \Omega^{2r}(N/\mathcal{G})$ . Moreover, if  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ , we have the Chern-Simons differential character  $\chi_{\mathfrak{A}} \in \hat{H}^{2r}((M \times N)/\mathcal{G})$  determined by the connection  $\Lambda_{\mathfrak{A}}$  which has curvature  $\underline{p}_{\mathfrak{A}}$ . As  $c: C \rightarrow M$  is a  $\mathcal{G}$ -invariant map, it induces a map  $c_N: (C \times N)/\mathcal{G} \rightarrow (M \times N)/\mathcal{G}$ . The character  $c_N^*(\chi_{\mathfrak{A}}) \in \hat{H}^{2r}((C \times N)/\mathcal{G})$  can be integrated over the fibre of  $(C \times N)/\mathcal{G} \rightarrow N/\mathcal{G}$  and we obtain a differential character  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^{2r-d}(N/\mathcal{G})$ . We have  $\text{curv}(\int_c \chi_{\mathfrak{A}}) = \int_c \underline{p}_{\mathfrak{A}} = \text{Cw}_{\mathfrak{A}}(\int_c p_{\mathcal{G}})$  and  $\text{char}(\int_c \chi_{\mathfrak{A}}) = \int_c \text{char}(\chi_{\mathfrak{A}})$ .

We are interested in the following cases

- 1) The action of  $\mathcal{G}$  on  $M$  and  $C$  is trivial (i.e.  $\mathcal{G}$  acts on  $P$  by gauge transformations). In this case the differential character  $\chi_{\mathfrak{A}}$  can be integrated over any map  $c: C \rightarrow M$ .
- 2)  $M$  is compact,  $\partial M = 0$  and  $\mathcal{G}$  preserves the orientation on  $M$ . In this case we can take  $c = \text{id}_M$  and we obtain a differential character  $\int_M \chi_{\mathfrak{A}} \in \hat{H}^{2r-d}(N/\mathcal{G})$  where  $d = \dim M$ .

3)  $M$  is a oriented manifold with compact boundary  $\partial M$  and  $\mathcal{G}$  preserves the orientation on  $M$ . In this case  $\chi_{\mathfrak{A}}$  can be integrated over  $c = \text{id}_{\partial M}$  and we obtain  $\int_{\partial M} \chi_{\mathfrak{A}} \in \hat{H}^{2r-d+1}(N/\mathcal{G})$ , where  $d = \dim M$ . Moreover, by equation (1) we have  $\int_{\partial M} \chi_{\mathfrak{A}} = \varsigma(\int_M \text{curv}(\chi_{\mathfrak{A}}))$ .

We call the characters  $\int_c \chi_{\mathfrak{A}}$  the integrated Chern-Simons characters. In this paper we study the geometrical interpretation of these characters of order 2 in Section 6 and of order 1 in Section 8. In the rest of this Section we give some technical results needed below.

## 5.2 Background connection

As it is commented in the Introduction, the equivariant prequantization bundles are given in terms of a background connection. Let  $A_0$  be a connection on  $P \rightarrow M$  (we call  $A_0$  a background connection) and we denote by  $\text{pr}_1 : P \times N \rightarrow P$  the projection. Then  $\mathbb{A}$  and  $\text{pr}_1^* A_0$  are connections on the same bundle  $P \times N \rightarrow M \times N$ , and hence we can define  $Tp(\mathbb{A}, \text{pr}_1^* A_0) \in \Omega^{2r-1}(M \times N)$ .

The product structure on  $M \times N$  induces a bigraduation  $\Omega^k(M \times N) \simeq \bigoplus_{i=0}^k \Omega^{i, k-i}(M \times N)$  with  $\Omega^{i, k-i}(M \times N) \simeq \Omega^i(M) \otimes \Omega^{k-i}(N)$ . Hence, we can decompose the curvature  $\mathbb{F} = \mathbb{F}^{2,0} + \mathbb{F}^{1,1} + \mathbb{F}^{0,2}$ . We have  $p(\mathbb{F}) = dTp(\mathbb{A}, \text{pr}_1^* A_0) + \text{pr}_1^* p(F_0)$ , with  $\text{pr}_1^* p(F_0) \in \Omega^{2r,0}(M \times N)$ . Hence for any  $u : U \rightarrow M$  with  $d = \dim U < 2r - 1$  we have

$$\int_u p(\mathbb{F}) = \int_u dTp(\mathbb{A}, \text{pr}_1^* A_0) = d \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) - (-1)^d \int_{\partial u} Tp(\mathbb{A}, \text{pr}_1^* A_0), \quad (2)$$

where we have used that  $\int_u \text{pr}_1^* p(F_0) = 0$  as  $\text{pr}_1^* p(F_0) \in \Omega^{2r,0}(M \times N)$ . If  $A'_0$  is another background connection and  $\eta(A_0, A'_0) = Tp(\mathbb{A}, \text{pr}_1^* A_0, \text{pr}_1^* A'_0)$  then we have  $Tp(\mathbb{A}, \text{pr}_1^* A'_0) = Tp(\mathbb{A}, \text{pr}_1^* A_0) + \text{pr}_1^* Tp(A_0, A'_0) + d\eta(A_0, A'_0)$ , with  $\text{pr}_1^* Tp(A_0, A'_0) \in \Omega^{2r-1,0}(M \times N)$ , and hence

$$\begin{aligned} \int_u Tp(\mathbb{A}, \text{pr}_1^* A'_0) &= \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) + \int_u \text{pr}_1^* Tp(A_0, A'_0) + \int_u d\eta(A_0, A'_0) \\ &= \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) + \int_u \text{pr}_1^* Tp(A_0, A'_0) + d \int_u \eta(A_0, A'_0) + (-1)^d \int_{\partial u} \eta(A_0, A'_0). \end{aligned}$$

If  $d = \dim U < 2r - 1$  we have  $\int_u \text{pr}_1^* Tp(A_0, A'_0) = 0$  as  $\text{pr}_1^* Tp(A_0, A'_0) \in \Omega^{2r-1,0}(M \times N)$  and hence

$$\int_u Tp(\mathbb{A}, \text{pr}_1^* A'_0) = \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) + d \int_u \eta(A_0, A'_0) + (-1)^d \int_{\partial u} \eta(A_0, A'_0). \quad (3)$$

Moreover, if  $\dim U = 2r - 1$  then  $\int_u \eta(A_0, A'_0) = 0$  and we have

$$\int_u Tp(\mathbb{A}, \text{pr}_1^* A'_0) = \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) + \int_u Tp(A_0, A'_0) - \int_{\partial u} \eta(A_0, A'_0). \quad (4)$$

## 5.3 Change of connection

We recall that we defined the integrated Chern-Simons characters using a connection  $\Lambda_{\mathfrak{A}}$  determined by the connection  $\mathbb{A}$  on  $\mathbb{P}$  and a connection  $\mathfrak{A}$  on

$N \rightarrow N/\mathcal{G}$ . If  $\mathfrak{A}'$  is another connection on  $N \rightarrow N/\mathcal{G}$  then we have  $\chi_{\mathfrak{A}'} = \chi_{\mathfrak{A}} + \varsigma(Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}}))$  and hence

$$\int_c \chi_{\mathfrak{A}'} = \int_c \chi_{\mathfrak{A}} + \varsigma(\int_c Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}})) = \int_c \chi_{\mathfrak{A}} + \varsigma(\int_c Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}})^{d, 2r-1-d}). \quad (5)$$

By Proposition 4 we have  $\pi^*Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}}) = r \int_0^1 p(\Lambda_{\mathfrak{A}'} - \Lambda_{\mathfrak{A}}, F_{\Lambda_t}, \dots, F_{\Lambda_t}) dt = r \int_0^1 p(v_{\mathbb{A}}(\mathfrak{A}' - \mathfrak{A}), \mathbb{F}_{\text{hor}_{\mathfrak{A}_t}} - \mathfrak{F}_t, \binom{r-1}{r-1}, \mathbb{F}_{\text{hor}_{\mathfrak{A}_t}} - \mathfrak{F}_t) dt$ , with  $\mathfrak{A}_t = t\mathfrak{A}' + (1-t)\mathfrak{A}$  and  $\mathfrak{F}_t$  is the curvature of  $\mathfrak{A}_t$ . Taking into account that  $\mathfrak{A}$  and  $\mathfrak{A}'$  come from a connection on  $N \rightarrow N/\mathcal{G}$  we have  $v_{\mathbb{A}}(\mathfrak{A}' - \mathfrak{A}) \in \Omega^{0,1}, (\mathbb{F}_{\text{hor}_{\mathfrak{A}_t}} - \mathfrak{F}_t)^{2,0} = \mathbb{F}^{2,0}$ . As a consequence we obtain the following conclusions

1) The component of bidegree  $(2r-1, 0)$  is  $Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}})^{2r-1,0} = 0$  and using this on equation (5) for  $d = 2r-1$  we conclude that  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^1(N/\mathcal{G})$  is independent of  $\mathfrak{A}$ .

2) The component of bidegree  $(2r-2, 1)$  is given by  $\pi^*Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}})^{2r-2,1} = -r \int_0^1 p(v_{\mathbb{A}}(\mathfrak{A}' - \mathfrak{A}), \mathbb{F}^{2,0}, \binom{r-1}{r-1}, \mathbb{F}^{2,0}) dt = -rp(v_{\mathbb{A}}(\mathfrak{A}' - \mathfrak{A}), \mathbb{F}^{2,0}, \binom{r-1}{r-1}, \mathbb{F}^{2,0})$ . If  $d = 2r-2$  then  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^2(N/\mathcal{G})$ , and  $\varpi_c = \int_c p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^2(N)$  can be written  $\varpi_c = \sigma_c + \mu_c$ , with  $\sigma_c = \int_c p(\mathbb{F})$  and  $\mu_c(X) = -r \int_c p(v_{\mathbb{A}}(X), \mathbb{F}, \binom{r-1}{r-1}, \mathbb{F}) = -r \int_c p(v_{\mathbb{A}}(X), \mathbb{F}^{2,0}, \binom{r-1}{r-1}, \mathbb{F}^{2,0})$ . Taking this into account we conclude that we have  $\pi^* \int_c Tp(\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}})^{2r-2,1} = -r \int_c p(v_{\mathbb{A}}(\mathfrak{A}' - \mathfrak{A}), \mathbb{F}^{2,0}, \binom{r-1}{r-1}, \mathbb{F}^{2,0}) = \mu_c(\mathfrak{A}' - \mathfrak{A})$ . In particular, using this on equation (5) it follows that for any curve  $\gamma$  on  $N$  such that  $\pi(\gamma(0)) = \pi(\gamma(1))$  we have the following equation that will be used in Section 6

$$(\int_c \chi_{\mathfrak{A}'})(\pi \circ \gamma) = (\int_c \chi_{\mathfrak{A}})(\pi \circ \gamma) + \int_{\gamma} \mu_c(\mathfrak{A}' - \mathfrak{A}). \quad (6)$$

## 5.4 Restriction to a subgroup

Let  $\mathcal{H} \subset \mathcal{G}$  be a subgroup of  $\mathcal{G}$ ,  $\mathfrak{A} \in \Omega^1(N, \text{Lie}\mathcal{G})$  a connection on  $\pi_{\mathcal{G}}: N \rightarrow N/\mathcal{G}$  and  $\mathfrak{A}' \in \Omega^1(N, \text{Lie}\mathcal{H})$  a connection on  $\pi_{\mathcal{H}}: N \rightarrow N/\mathcal{H}$ . We recall that by Proposition 4 we have  $\Lambda_{\mathfrak{A}'}, \Lambda_{\mathfrak{A}} \in \Omega^1(P \times N, \mathfrak{g})$  with  $\Lambda_{\mathfrak{A}}$   $\mathcal{G}$ -projectable and  $\Lambda_{\mathfrak{A}'}$   $\mathcal{H}$ -projectable. As  $\mathfrak{A}' \in \Omega^1(N, \text{Lie}\mathcal{H}) \subset \Omega^1(N, \text{Lie}\mathcal{G})$  we can consider that  $\mathfrak{A}' - \mathfrak{A} \in \Omega^1(N, \text{Lie}\mathcal{G})$ . If  $c: C \rightarrow N$  is a  $\mathcal{G}$ -equivariant map we have  $\int_c \chi_{\mathfrak{A}'} \in \hat{H}^2(N/\mathcal{H})$  and  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^2(N/\mathcal{G})$ . We can repeat the arguments of the preceding section to show that for any curve  $\gamma$  on  $N$  such that  $\pi_{\mathcal{H}}(\gamma(0)) = \pi_{\mathcal{H}}(\gamma(1))$  we have

$$(\int_c \chi_{\mathfrak{A}'})(\pi_{\mathcal{H}} \circ \gamma) = (\int_c \chi_{\mathfrak{A}})(\pi_{\mathcal{G}} \circ \gamma) + \int_{\gamma} \mu_c(\mathfrak{A}' - \mathfrak{A}). \quad (7)$$

## 5.5 Naturality of integrated Chern-Simons characters

Suppose that we have the data of Section 5.1, and that  $\mathcal{G}$  acts on another bundle  $P' \rightarrow M'$  by  $G$ -bundle automorphisms,  $N' \rightarrow N'/\mathcal{G}$  is another principal  $\mathcal{G}$ -bundle,  $f: P' \rightarrow P$  is a  $\mathcal{G}$ -equivariant morphism of principal  $G$ -bundles and  $g: N' \rightarrow N$  is a  $\mathcal{G}$ -equivariant map. Then  $\mathbb{A}' = (f \times g)^*\mathbb{A}$  is a  $\mathcal{G}$ -invariant connection on  $\mathbb{P}' = P' \times N' \rightarrow M' \times N'$ ,  $\mathfrak{A}' = g^*\mathfrak{A}$  is a connection on  $N' \rightarrow N'/\mathcal{G}$  and we have the Chern-Simons differential character  $\chi_{\mathfrak{A}'}$ . Using the naturality of Chern-Simons characters, the definition of  $\chi_{\mathfrak{A}}$  and Proposition 4 we obtain

$\chi_{\mathfrak{A}'} = (\underline{f} \times g)^* \chi_{\mathfrak{A}}$ . Furthermore, if  $\mathcal{G}$  acts on  $C$ , and  $c: C \rightarrow M$ ,  $c': C \rightarrow M'$  are  $\mathcal{G}$ -equivariant maps such that  $c = \underline{f} \circ c'$ , by the definition of the integrated Chern-Simons characters we have  $\int_{c'} \chi_{\mathfrak{A}'} = g^*(\int_c \chi_{\mathfrak{A}})$ .

## 6 Second order integrated Chern-Simons characters

Assume that we have the data of Section 5.1 and  $\dim C = d = 2r - 2$ . Then we have  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^2(N/\mathcal{G})$  and  $\text{curv}(\int_c \chi_{\mathfrak{A}}) = \int_c \underline{p}_{\mathfrak{A}} = \text{Cw}_{\mathfrak{A}}(\int_c p_{\mathcal{G}}) \in \Omega_{\mathcal{G}}^2(N/\mathcal{G})$ . By Section 3.2  $\varpi_c = \int_c p_{\mathcal{G}}$  can be written  $\varpi_c = \sigma_c + \mu_c$ , and by the definition of  $p_{\mathcal{G}}$  we have  $\sigma_c = \int_c p(\mathbb{F}, \dots, \mathbb{F})$  and  $\mu_c(X) = -r \int_c p(v_{\mathbb{A}}(X), \mathbb{F}, \overset{r-1}{\cdot}, \mathbb{F})$ . If  $A_0$  is a background connection we define  $\rho_c = \int_c Tp(\mathbb{A}, \text{pr}_1^* A_0) \in \Omega^1(N)$ , and by equation (2) we have  $\sigma_c = d\rho_c$ . If we set  $\lambda_c = \rho_c + \mu_c(\mathfrak{A})$ , then by Lemma 3 we have  $\pi^*(\int_c \underline{p}_{\mathfrak{A}}) = d\lambda_c$ . By applying Proposition 5 we obtain the following

**Proposition 13** *Let  $A_0$  be a background connection on  $P \rightarrow M$  and  $\mathfrak{A}$  be a connection on  $N \rightarrow N/\mathcal{G}$ . Then there exists a unique lift of the action of  $\mathcal{G}$  on  $N$  to an action on  $\mathcal{U}_c = N \times U(1)$  by  $U(1)$ -bundle automorphisms such that  $\Theta_c = \theta - 2\pi i(\rho_c + \mu_c(\mathfrak{A})) \in \Omega^1(N \times U(1), i\mathbb{R})$  is projectable onto a connection  $\underline{\Theta}_c$  on  $\underline{\mathcal{U}}_c = (N \times U(1))/\mathcal{G} \rightarrow N/\mathcal{G}$  and  $\int_c \chi_{\mathfrak{A}} = \log \text{hol}_{\underline{\Theta}_c}$ . The action of  $\phi \in \mathcal{G}$  on  $N \times U(1)$  is given by  $\Phi: N \times U(1) \rightarrow N \times U(1)$ ,  $\Phi(x, u) = (\phi x, \exp(2\pi i \alpha_{\phi}(x)) \cdot u)$ , where  $\alpha_{\phi}: N \rightarrow \mathbb{R}/\mathbb{Z}$  is defined by  $\alpha_{\phi}(x) = \int_{\gamma}(\rho_c + \mu_c(\mathfrak{A})) - (\int_c \chi_{\mathfrak{A}})(\pi \circ \gamma)$ , and  $\gamma$  is any curve on  $N$  joining  $x$  and  $\phi x$ .*

In particular  $(\underline{\mathcal{U}}_c, \underline{\Theta}_c)$  is a prequantization bundle of  $(N/\mathcal{G}, \int_c \underline{p}_{\mathfrak{A}})$ . We recall that a  $\mathcal{G}$ -equivariant section of  $\mathcal{U}_c \rightarrow N$  is determined by a map  $\underline{S}: N \rightarrow U(1)$   $S(x) = \exp(2\pi i \cdot s(x))$  where  $s: N \rightarrow \mathbb{R}/\mathbb{Z}$  satisfies  $\alpha_{\phi}(x) = s(\phi x) - s(x)$ . It determines a section of the quotient bundle  $\underline{S}: N/\mathcal{G} \rightarrow \mathcal{U}/\mathcal{G}$ . The following result shows that our bundle generalizes the Chern-Simons line

**Proposition 14** *If  $c = \partial u$  for a  $\mathcal{G}$ -equivariant map  $u: U \rightarrow M$  and we define  $s_u = -\int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) \in \Omega^0(N)$ , then  $\alpha_{c, \phi}(A) = s_u(\phi x) - s_u(x)$ . Hence  $S_u = \exp(2\pi i \cdot s_u)$  determines a section of  $\mathcal{U}_c/\mathcal{G} \rightarrow N/\mathcal{G}$ .*

*Furthermore, the covariant derivative of  $S_u$  is  $\nabla^{\underline{\Theta}_c} S_u = -2\pi i \int_u \underline{p}_{\mathfrak{A}} \cdot S_u$ .*

**Proof.** If we define  $\sigma_u = \int_u p_{\mathcal{G}} = \int_u p(\mathbb{F})$  then  $D\sigma_u = D(\int_u p_{\mathcal{G}}) = \int_u (Dp_{\mathcal{G}}) + \int_{\partial u} p_{\mathcal{G}} = \int_c p_{\mathcal{G}} = \varpi_c = \sigma_c + \mu_c$ . Hence  $d\sigma_u = \sigma_c$  and  $\iota_{X_N} \sigma_u = -\mu_c(X)$ . In particular  $\sigma_u(\mathfrak{A}) = -\mu_c(\mathfrak{A})$  for any  $Y \in TN$ . Using this equation and the results of Section 3.1 we have  $\pi^*(\int_u \underline{p}_{\mathfrak{A}}) = \pi^*(\int_u \text{Cw}_{\mathfrak{A}}(p_{\mathcal{G}})) = \pi^*(\text{Cw}_{\mathfrak{A}}(\int_u p_{\mathcal{G}})) = \pi^*(\text{Cw}_{\mathfrak{A}}(\sigma_u)) = \sigma_u + \mu_c(\mathfrak{A})$ . Moreover by equation (2) we have  $\sigma_u = \int_u p(\mathbb{F}) = d \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0) + \int_{\partial u} Tp(\mathbb{A}, \text{pr}_1^* A_0) = -ds_u + \rho_c = -ds_u + \lambda_c - \mu_c(\mathfrak{A})$ . For  $\gamma$  a curve joining  $x$  and  $\phi x$ , using the preceding equations and equation (1) we have

$$\begin{aligned} \alpha_{\partial u, \phi}(x) &= \int_{\gamma} \lambda_c - \left( \int_{\partial u} \chi_{\mathfrak{A}} \right) (\pi \circ \gamma) = \int_{\gamma} (\sigma_u + ds_u + \mu_c(\mathfrak{A})) - \int_{(\pi \circ \gamma)} \int_u \underline{p}_{\mathfrak{A}} \\ &= \int_{\gamma} (\sigma_u + ds_u + \mu_c(\mathfrak{A})) - \int_{\gamma} (\sigma_u + \mu_c(\mathfrak{A})) = \int_{\gamma} ds_u = \int_{\partial \gamma} s_u = s_u(\phi x) - s_u(x). \end{aligned}$$

Finally we have  $\pi^*(\nabla^\Theta S_u) = dS_u - 2\pi i \lambda_c \cdot S_u = 2\pi i ds_u \cdot S_u - 2\pi i \lambda_c \cdot S_u = -2\pi i(\sigma_u + \mu_c(\mathfrak{A})) \cdot S_u = -\pi^*(2\pi i(\int_u \underline{p}_{\mathfrak{A}}) \cdot S_u)$ . ■

**Remark 15** We can also consider that  $\exp(2\pi i \cdot \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0))$  determines a section of the inverse bundle  $\mathcal{U}_c^{-1}$  as it is done in [14].

A fundamental property of the integrated Chern-Simons, is the following

**Proposition 16** *The cocycle  $\alpha$  is independent of the connection  $\mathfrak{A}$  chosen in  $N \rightarrow N/\mathcal{G}$ .*

**Proof.** If  $\mathfrak{A}'$  is another connection on  $\pi_{\mathcal{G}}: N \rightarrow N/\mathcal{G}$  we have  $\lambda'_c = \rho_c + \mu_c(\mathfrak{A}') = \lambda_c + \mu_c(\mathfrak{A}' - \mathfrak{A})$  and by equation (6) for any curve  $\gamma$  on  $N$  such that  $\pi_{\mathcal{G}}(\gamma(0)) = \pi_{\mathcal{G}}(\gamma(1))$  we have  $\int_c \chi_{\mathfrak{A}'}(\pi_{\mathcal{G}} \circ \gamma) = \int_c \chi_{\mathfrak{A}}(\pi_{\mathcal{G}} \circ \gamma) + \int_{\gamma} \mu_c(\mathfrak{A}' - \mathfrak{A})$ . Hence  $\alpha'_{\phi}(A) = \int_{\gamma} \lambda' - \int_c \chi_{\mathfrak{A}'}(\pi_{\mathcal{G}} \circ \gamma) = \int_{\gamma} \lambda + \int_{\gamma} \mu_c(\mathfrak{A}' - \mathfrak{A}) - \int_c \chi_{\mathfrak{A}}(\pi_{\mathcal{G}} \circ \gamma) - \int_{\gamma} \mu_c(\mathfrak{A}' - \mathfrak{A}) = \alpha_{\phi}(A)$ . ■

As the action does not depend on  $\mathfrak{A}$ , we give a version of Proposition 13 which is independent of  $\mathfrak{A}$ , and that (as we show in the next Section) can be generalized to non free actions. The connection  $\Theta_c$  depends on  $\mathfrak{A}$ , and hence we need to choose another connection. If we define  $\Xi_c = \theta - 2\pi i \rho_c = \Theta_c + 2\pi i \mu_c(\mathfrak{A}) \in \Omega^1(N \times U(1), i\mathbb{R})$  then  $\Xi_c$  does not depend on  $\mathfrak{A}$ . By the properties of  $\Theta_c$  we conclude that  $\Xi_c$  is  $\mathcal{G}$ -invariant and we have  $D\Xi_c(X) = -2\pi i d\rho_c - \iota_{X_{N \times U(1)}} \Xi_c = -2\pi i(\sigma_c + \mu_c(X)) = -2\pi i \varpi_c$ . We also have  $c_1(\mathcal{U}_c/\mathcal{G}) = \text{char}(\int_c \chi_{\mathfrak{A}}) = \int_c \text{char}(\chi_{\mathfrak{A}}) = \int_c \Upsilon_{\mathbb{P}/\mathcal{G}} \in H^2(N/\mathcal{G}, \mathbb{Z})$ .

The lift of a vector field is given by  $X_{N \times U(1)} = X_N + 2\pi i \cdot \lambda_c(X_N) \partial_{\theta} = X_N + 2\pi i \cdot (\rho_c(X_N) + \mu_c(\mathfrak{A})(X_N)) \partial_{\theta} = X_N + 2\pi i \cdot (\rho_c(X_N) + \mu_c(X)) \partial_{\theta}$ , (note that  $X_{N \times U(1)}$  does not depend on  $\mathfrak{A}$ ). If  $\phi \in \mathcal{G}_0$  we have  $\alpha_{c,\phi}(x) = \int_{\varphi x} \lambda_c = \int_{\varphi x} \rho_c + \int_{\varphi x} \mu_c(\mathfrak{A})$ , where  $\varphi: I \rightarrow \mathcal{G}$  is a curve with  $\varphi_0 = 1_{\mathcal{G}}$  and  $\varphi_1 = \phi$ . We can give a version of this formula independent of  $\mathfrak{A}$  as follows. If  $\varphi: I \rightarrow \mathcal{G}$ , we define  $\dot{\varphi}: I \rightarrow \text{Lie}\mathcal{G}$  by  $\dot{\varphi}_t = R_{\varphi_t^{-1}} \dot{\varphi}$ . The form  $\mu_c(\mathfrak{A})$  depends on  $\mathfrak{A}$ , but  $\int_{\varphi x} \mu_c(\mathfrak{A}) = \int_I (\mu_c)_{\varphi_t x}(\dot{\varphi}_t) dt$ . Hence, if we define  $\oint_{\varphi x} \mu_c = \int_I (\mu_c)_{\varphi_t x}(\dot{\varphi}_t) dt$  then we have  $\alpha_{c,\phi}(x) = \int_{\varphi x} \lambda_c = \int_{\varphi x} \rho_c + \oint_{\varphi x} \mu_c$ , which does not depend on  $\mathfrak{A}$ .

Finally, if  $c = \partial u$  we have  $\nabla^\Xi S_u = \nabla^\Theta S_u + 2\pi i \mu_c(\mathfrak{A}) = -2\pi i(\sigma_u + \mu_c(\mathfrak{A}) - \mu_c(\mathfrak{A})) \cdot S_u = -2\pi i \sigma_u \cdot S_u$ . Thus we have proved the following equivariant version of Proposition 13

**Theorem 17** *Let  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ ,  $A_0$  a background connection on  $P$ . and  $c: C \rightarrow M$  a  $\mathcal{G}$ -invariant map. These data determine an action of  $\mathcal{G}$  on  $\mathcal{U}_c = N \times U(1) \rightarrow N$  by  $U(1)$ -bundle automorphisms  $\Phi(x, u) = (\phi x, \exp(2\pi i \alpha_{c,\phi}(x)) \cdot u)$  such that*

i) *the connection  $\Xi_c = \theta - 2\pi i \rho_c$  is  $\mathcal{G}$ -invariant, and the  $\mathcal{G}$ -equivariant curvature of  $\Xi_c$  is  $\varpi_c$ .*

ii)  $c_1(\mathcal{U}_c/\mathcal{G}) = \int_c \Upsilon_{\mathbb{P}/\mathcal{G}} \in H^2(N/\mathcal{G}, \mathbb{Z})$ .

*For every  $X \in \text{Lie}\mathcal{G}$  we have  $X_{N \times U(1)} = X_N + 2\pi i(\rho_c(X_N) + \mu_c(X)) \partial_{\theta}$ . If  $\phi \in \mathcal{G}_0$  then  $\alpha_{c,\phi}(x) = \int_{\varphi x} \rho_c + \oint_{\varphi x} \mu_c$ , where  $\varphi$  is any curve on  $\mathcal{G}_0$  joining  $1_{\mathcal{G}}$  and  $\phi$ .*



If  $c = \partial u$  for a  $\mathcal{G}$ -equivariant map  $u: U \rightarrow M$  and  $s_u = -\int_u Tp(\mathbb{A}, \text{pr}_1^* A_0)$ , then  $\alpha_{c,\phi}(x) = s_u(\phi x) - s_u(x)$ . Hence  $S_u = \exp(2\pi i \cdot s_u)$  determines a  $\mathcal{G}$ -equivariant section of  $\mathcal{U}_c \rightarrow N$ . Moreover, we have  $\nabla^{\Xi_c} S_u = -2\pi i \sigma_u \cdot S_u$ , where  $\sigma_u = \int_u p(\mathbb{F}) \in \Omega^1(N)$ .

**Remark 18** If  $\mathcal{H}$  is a subgroup of  $\mathcal{G}$ , for  $\phi \in \mathcal{H}$  in theory we have two different actions  $\Phi^{\mathcal{H}}$  and  $\Phi^{\mathcal{G}}$  of  $\phi$  on  $\mathcal{U}_c$ . However, it can be shown that  $\Phi^{\mathcal{H}} = \Phi^{\mathcal{G}}$  in a similar way to the proof of Proposition 16, but using equation 7 instead of equation 6.

We make some comments about the unicity of the action. In Proposition 13 the differential character  $\int_c \chi_{\mathfrak{A}}$  and the background connection  $A_0$  determine a unique action  $\alpha$ . In Theorem 17 we replace  $\int_c \chi_{\mathfrak{A}}$  by the equivariant curvature  $\varpi$  and the characteristic class  $c_1(\mathcal{U}_c/\mathcal{G})$ . As it is well known (see e.g. [5, 11]) the curvature and characteristic class do not determine a differential character, and hence the unicity of the action is not garantized. In more detail, if  $\alpha$  and  $\alpha'$  are two actions satisfying condition i) then it can be seen that  $\kappa_\phi = \alpha_\phi - \alpha'_\phi$  determines an element  $\kappa \in \text{Hom}(\mathcal{G}/\mathcal{G}_0, \mathbb{R}/\mathbb{Z})$ . If condition ii) is also satisfied, then it can be proved that  $\kappa \in \text{Hom}(\mathcal{G}/\mathcal{G}_0, \mathbb{R})$  and  $\alpha = \alpha'$  if  $\kappa \in \text{Hom}(\mathcal{G}/\mathcal{G}_0, \mathbb{Z})$ . In general it could be possible that  $\alpha \neq \alpha'$ , but under certain circumstances it is not possible. For example, if  $\mathcal{G}/\mathcal{G}_0$  is a torsion group (i.e. all elements are torsion) then  $\kappa = 0$  and the action is uniquely determined by conditions i) and ii). In particular this happens if  $\mathcal{G}$  has a finite number of connected components.

**Remark 19** In place of the principal  $U(1)$ -bundle  $\mathcal{U}_c$ , we can consider the  $\mathcal{G}$ -equivariant Hermitian line bundle  $\mathcal{L}_c = N \times \mathbb{C} \rightarrow N$  with the action  $\Phi(x, z) = (\phi x, \exp(2\pi i \alpha_{c,\phi}(x)) \cdot z)$  and  $\Xi_c$  determines a hermitian connection  $\nabla^{\Xi_c}$  on this bundle with  $\nabla^{\Xi_c} f = df - 2\pi i \rho_c \cdot f$  for  $f: N \rightarrow \mathbb{C}$ .

## 6.1 Change of background connection

The prequantization bundle  $\mathcal{U}_c$  and the connection  $\Xi_c$  are defined using a background connection  $A_0$ . The next Proposition shows that the action changes under a change of  $A_0$ , but the corresponding prequantization bundles are isomorphic.

**Proposition 20** Let  $A'_0$  be another background connection and denote by  $\mathcal{U}'_c, \Xi'_c$  and  $\alpha'_c$  the bundle, connection and action determined by  $A'_0$ . If we define  $\beta_c = \int_c Tp(\mathbb{A}, \text{pr}_1^* A_0, \text{pr}_1^* A'_0) \in \Omega^0(N)$  then  $\Xi'_c = \Xi_c - 2\pi i d\beta_c$  and  $\alpha'_{c,\phi} = \alpha_{c,\phi} + \phi^* \beta - \beta$ . The map  $\Psi: \mathcal{U}_c \rightarrow \mathcal{U}'_c$   $\Psi(x, u) = (x, \exp(2\pi i \beta_c(x)) \cdot u)$  is a  $\mathcal{G}$ -equivariant isomorphism of  $U(1)$ -bundles and  $\Psi^*(\Xi'_c) = \Xi_c$ .

**Proof.** It follows easily from the definitions and the equality  $\rho'_c = \rho_c + d\beta_c$ , which is a consequence of equation (3). ■

**Remark 21** We interpret this result in the following way.  $(p, \Upsilon) \in I_{\mathbb{Z}}^r(G)$  and  $c: C \rightarrow M$  determine a  $\mathcal{G}$ -equivariant prequantization bundle  $(\mathcal{U}_c, \Xi_c)$  for  $(N, \varpi_c)$ , and a background connection  $A_0$  determines a global trivialization of this bundle. In this sense, the prequantization bundle does not depend on  $A_0$ .

The situation is different for the section associated to the Chern-Simons action, as using equation (4) we obtain the following

**Proposition 22** *If  $c = \partial u$  for a  $\mathcal{G}$ -equivariant map  $u: U \rightarrow M$  and  $S_u, S'_u$  are the sections associated to  $A_0$  and  $A'_0$  then  $\Psi \circ S_u = S'_u \cdot \exp(2\pi i \int_u Tp(A_0, A'_0))$ .*

Hence the section  $S_u$  is not intrinsically determined by  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$  and  $u$ . To explain this, note that if  $S$  is a section satisfying  $\nabla^{\Xi_c} S = -2\pi i \sigma_u \cdot S$ , any other section satisfying this condition is given by  $\exp(ia)S$  for  $a \in \mathbb{R}$  constant. The background connection  $A_0$  determines a constant  $a$  and another connection  $A'_0$  determines a different constant  $a'$ , and hence a different section.

## 6.2 Extension to non-free actions

Now we assume that we have the data of Section 5.1 satisfying a), b) and c) but not necessarily d), and that  $\dim C = d = 2r - 2$ . Let  $E$  be a connected and simply connected manifold in which  $\mathcal{G}$  acts and such that the action of  $\mathcal{G}$  on the product  $\overline{N} = N \times E$  is free and  $N \times E \rightarrow (N \times E)/\mathcal{G}$  is a principal  $\mathcal{G}$ -bundle. For example we can take  $E = E\mathcal{G}$ , or a simpler finite dimensional space. We denote by  $\text{pr} = M \times \overline{N} \rightarrow M \times N$  and  $\overline{\text{pr}}: \overline{\mathbb{P}} = \mathbb{P} \times E \rightarrow \mathbb{P}$  the projections, which are  $\mathcal{G}$ -invariant maps. Hence  $\overline{\mathbb{A}} = q^*\mathbb{A}$  is a  $\mathcal{G}$ -invariant connection on  $\overline{\mathbb{P}} \rightarrow M \times N \times E$ . If we apply Theorem 17 to  $\overline{N}$ ,  $\overline{\mathbb{A}}$  and  $\overline{\mathbb{P}}$  we obtain a cocycle  $\bar{\alpha}_c: \mathcal{G} \times \overline{N} \rightarrow \mathbb{R}/\mathbb{Z}$  and a connection  $\bar{\Xi}_c = \theta - 2\pi i \bar{\rho}_c$  with  $\bar{\rho}_c = \int_c Tp(\overline{\mathbb{A}}, \overline{\text{pr}}^* \text{pr}_1^* A_0) = \text{pr}^*(\int_c Tp(\mathbb{A}, \text{pr}_1^* A_0)) = \text{pr}^* \rho_c$ .

**Lemma 23** *We have*

- a)  $\bar{\alpha}_{c, \phi}(x, e)$  does not depend on  $e \in E$ , and hence  $\bar{\alpha}_c = \text{pr}^* \alpha_c$  for a cocycle  $\alpha_c: \mathcal{G} \times N \rightarrow \mathbb{R}/\mathbb{Z}$ .
- b)  $\alpha_c$  does not depend of the manifold  $E$  chosen.

**Proof.** a) Recall that if  $\gamma_1$  is a curve on  $N$  joining  $x$  and  $\phi x$  and  $\gamma_2$  a curve on  $E$  joining  $e$  and  $\phi e$  we have  $\bar{\alpha}_{c, \phi}(x, e) = \int_{\gamma_1 \times \gamma_2} \bar{\lambda}_c - \bar{\chi}(\gamma_1 \times \gamma_2)$ , where  $\bar{\chi} = \int_c \chi_{\mathfrak{A}} \in \hat{H}^2(\overline{N}/\mathcal{G})$  is the integrated Chern-Simons character associated to a connection  $\mathfrak{A}$  on  $\overline{N} \rightarrow \overline{N}/\mathcal{G}$  and  $\bar{\lambda}_c = \bar{\rho}_c + \bar{\mu}_c(\mathfrak{A})$ . By Lemma 7, if  $e'$  is another point on  $E$  and  $\gamma_{ee'}$  a curve on  $E$  joining  $e$  and  $e'$  (it exists as  $E$  is connected) we have  $\alpha_{\phi}(x, e') = \alpha_{\phi}(x, e) + \int_{\{x\} \times \gamma_{ee'}} (\phi^* \bar{\lambda}_c - \bar{\lambda}_c)$ . But  $\phi^* \bar{\lambda}_c - \bar{\lambda}_c = \phi^* \bar{\rho}_c - \bar{\rho}_c = \text{pr}^*(\phi^* \rho_c - \rho_c)$  and hence  $\int_{\{x\} \times \gamma_{ee'}} (\phi^* \bar{\lambda}_c - \bar{\lambda}_c) = \int_{\{x\} \times \gamma_{ee'}} \text{pr}^*(\phi^* \rho_c - \rho_c) = 0$ .

b) Let  $E_1, E_2$  be two connected and simply connected manifolds and we denote by  $\alpha_1$  and  $\alpha_2$  the associated actions. Then  $E = E_1 \times E_2$  is also connected and simply connected. We denote by  $\alpha_{12}$  the function corresponding to  $E$  and by  $q_i: E \rightarrow E_i$  the projections. By the naturality of the integrated Chern-Simons characters (see Section 5.5) and the definition of  $\alpha$ , we have  $\alpha_{12} = q_1^* \alpha_1$  and  $\alpha_{12} = q_2^* \alpha_2$  and hence  $\alpha_1(x) = \bar{\alpha}_1(x, e_1) = \bar{\alpha}_{12}(x, e_1, e_2) = \bar{\alpha}_2(x, e_2) = \alpha_2(x)$  for any  $(x, e_1, e_2) \in N \times E_1 \times E_2$ . ■

Taking  $E = E\mathcal{G}$  we obtain  $c_{1, \mathcal{G}}(\mathcal{U}) = c_1(\mathcal{U}_{\mathcal{G}}) = \int_c \Upsilon_{\mathbb{P}_{\mathcal{G}}} = \int_c \Upsilon_{\mathbb{P}} \in H^2(N_{\mathcal{G}}, \mathbb{Z})$ , where  $\Upsilon_{\mathbb{P}} \in H^{2r}((M \times N)_{\mathcal{G}}, \mathbb{Z})$  is the  $\mathcal{G}$ -equivariant characteristic class of  $\mathbb{P}$

associated to  $\Upsilon$ . Hence we obtain an action of  $\mathcal{G}$  on  $N \times U(1)$  by  $U(1)$ -bundle automorphisms, i.e., a  $\mathcal{G}$ -equivariant  $U(1)$ -bundle on  $N$ . We conclude that Theorem 17 and Proposition 20 are valid for any action, with the exception of condition ii) that should be replaced by the equivariant version

$$\text{ii}^*) \quad c_{1,\mathcal{G}}(\mathcal{U}_c) = \int_c \Upsilon_{\mathbb{P}} \in H^2(N_{\mathcal{G}}, \mathbb{Z}).$$

**Remark 24** By Remark 18 to define the action of  $\phi \in \mathcal{G}$ , we can consider any Lie subgroup  $\mathcal{H} \subset \mathcal{G}$  such that  $\phi \in \mathcal{H}$  and  $E = E\mathcal{H}$ . This could be useful for example when  $\mathcal{G}$  is an infinite dimensional Lie group. In place of working  $E\mathcal{G}$  we can work with the universal bundle of a finite dimensional subgroup.

### 6.3 Change of polynomial and submanifold

The action of  $\mathcal{G}$  on  $\mathcal{A} \times U(1)$  is defined by a map  $\Phi_{\alpha}(x, u) = (\phi x, \exp(2\pi i \alpha_{\phi}(x)) \cdot u)$  where  $\alpha: \mathcal{G} \times N \rightarrow \mathbb{R}/\mathbb{Z}$  satisfies the cocycle condition  $\alpha_{\phi_2 \phi_1}(x) = \alpha_{\phi_1}(x) + \alpha_{\phi_2}(\phi_1 x)$ . If  $\alpha$  and  $\alpha'$  satisfy the cocycle condition, then it is also satisfied by  $-\alpha$  and  $\alpha + \alpha'$ , and  $\Phi_{-\alpha} = \Phi_{\alpha}^{-1}$  and  $\Phi_{\alpha+\alpha'} = \Phi_{\alpha} \cdot \Phi_{\alpha'}$ . In terms of line bundles, if  $\mathcal{L}^{\alpha}$  is the  $\mathcal{G}$ -equivariant line bundle associated to a cocycle  $\alpha$ , then  $\Phi_{-\alpha}$  corresponds to the dual bundle  $\mathcal{L}^{-\alpha} = (\mathcal{L}^{\alpha})^*$  and  $\Phi_{\alpha+\alpha'}$  corresponds to the tensor product  $\mathcal{L}^{\alpha+\alpha'} = \mathcal{L}^{\alpha} \otimes \mathcal{L}^{\alpha'}$ .

We denote by  $\alpha_c^{\vec{p}}$  the action determined by  $\vec{p} = (p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ ,  $c: C \rightarrow N$  and by  $(\mathcal{L}_c^{\vec{p}}, \nabla_c^{\vec{p}})$  the  $\mathcal{G}$ -equivariant line bundle and connection determined by them. If  $c: C \rightarrow N$ ,  $c': C' \rightarrow N$  are two smooth maps we define  $-c: -C \rightarrow N$ , where  $-C$  is the manifold  $C$  with the opposite orientation and  $c+c': C \sqcup C' \rightarrow N$ . Then we have  $\alpha_{-c}^{\vec{p}} = -\alpha_c^{\vec{p}}$ ,  $\alpha_{c+c'}^{\vec{p}} = \alpha_c^{\vec{p}} + \alpha_{c'}^{\vec{p}}$  and also  $\rho_{-c}^{\vec{p}} = -\rho_c^{\vec{p}}$ ,  $\rho_{c+c'}^{\vec{p}} = \rho_c^{\vec{p}} + \rho_{c'}^{\vec{p}}$ . We conclude that  $(\mathcal{L}_{-c}^{\vec{p}}, \nabla_{-c}^{\vec{p}}) = ((\mathcal{L}_c^{\vec{p}}, \nabla_c^{\vec{p}}))^*$  and  $(\mathcal{L}_{c+c'}^{\vec{p}}, \nabla_{c+c'}^{\vec{p}}) = (\mathcal{L}_c^{\vec{p}}, \nabla_c^{\vec{p}}) \otimes (\mathcal{L}_{c'}^{\vec{p}}, \nabla_{c'}^{\vec{p}})$ .

In a similar way if  $\vec{p} = (p, \Upsilon)$ ,  $\vec{p}' = (p', \Upsilon') \in \mathcal{I}_{\mathbb{Z}}^r(G)$  then we have  $\alpha_c^{-\vec{p}} = -\alpha_c^{\vec{p}}$ ,  $\alpha_c^{\vec{p}+\vec{p}'} = \alpha_c^{\vec{p}} + \alpha_c^{\vec{p}'}$  and  $\rho_c^{-\vec{p}} = -\rho_c^{\vec{p}}$ ,  $\rho_c^{\vec{p}+\vec{p}'} = \rho_c^{\vec{p}} + \rho_c^{\vec{p}'}$ . Hence  $(\mathcal{L}_c^{-\vec{p}}, \nabla_c^{-\vec{p}}) = ((\mathcal{L}_c^{\vec{p}}, \nabla_c^{\vec{p}}))^*$  and  $(\mathcal{L}_c^{\vec{p}+\vec{p}'}, \nabla_c^{\vec{p}+\vec{p}'}) = (\mathcal{L}_c^{\vec{p}}, \nabla_c^{\vec{p}}) \otimes (\mathcal{L}_c^{\vec{p}'}, \nabla_c^{\vec{p}'})$ .

If  $\partial u = c - c'$ , by Theorem 17  $S_u = \exp(-2\pi i \cdot \int_u Tp(\mathbb{A}, \text{pr}_1^* A_0))$  determines a  $\mathcal{G}$ -equivariant section of unitary norm of  $\mathcal{L}_{c-c'}^{\vec{p}} \simeq \mathcal{L}_c^{\vec{p}} \otimes (\mathcal{L}_{c'}^{\vec{p}})^* \simeq \text{Hom}(\mathcal{L}_{c'}^{\vec{p}}, \mathcal{L}_c^{\vec{p}})$  and hence  $\mathcal{L}_{c'}^{\vec{p}}$  and  $\mathcal{L}_c^{\vec{p}}$  are isomorphic as  $\mathcal{G}$ -equivariant line bundles.

**Remark 25** It is important to recall that in the preceding formulas we are using the same background connection  $A_0$  (i.e. the same trivialization (see Remark 21)) for all the bundles. If we use different connections  $A_0$  and  $A'_0$  for  $c$  and  $-c$ , we do not have  $\mathcal{L}_c^{A_0} = (\mathcal{L}_{-c}^{A'_0})^*$ , but we have a pairing  $\mathcal{L}_c^{A_0} \otimes (\mathcal{L}_{-c}^{A'_0})^* \rightarrow N \times \mathbb{C}$ .

## 7 The space of connections

In this section the constructions of Section 6 are applied to the space of connections on a principal bundle. Let  $P \rightarrow M$  be a principal  $G$ -bundle, and  $\mathcal{A}$  the space of principal connections on this bundle. As  $\mathcal{A}$  is an affine space modeled on  $\Omega^1(M, \text{ad}P)$ , we have canonical isomorphisms  $T_A \mathcal{A} \simeq \Omega^1(M, \text{ad}P)$  for

any  $A \in \mathcal{A}$ . The Lie algebra of  $\text{Aut}P$  is the space of  $G$ -invariant vector fields on  $P$ ,  $\text{aut}P \subset \mathfrak{X}(P)$ , and the Lie algebra of  $\text{Gau}P$  is the subspace of vertical  $G$ -invariant vector fields. We have an identification  $\text{gau}P \simeq \Omega^0(M, \text{ad}P)$ . The group  $\text{Aut}P$  acts on  $\mathcal{A}$  and for any  $X \in \text{aut}P$  we have  $X_{\mathcal{A}}(A) = d^A(v_A(X))$ . In particular, if  $X \in \text{gau}P \simeq \Omega^0(M, \text{ad}P)$  we have  $v_A(X) \simeq X$  and  $X_{\mathcal{A}}(A) = d^A X$ . The principal  $G$ -bundle  $\mathbb{P} = P \times \mathcal{A} \rightarrow M \times \mathcal{A}$  has a tautological connection  $\mathbb{A} \in \Omega^1(P \times \mathcal{A}, \mathfrak{g})$  defined by  $\mathbb{A}_{(x,A)}(X, Y) = A_x(X)$  for  $(x, A) \in P \times \mathcal{A}$ ,  $X \in T_x P$ ,  $Y \in T_A \mathcal{A}$ . We denote by  $\mathbb{F}$  the curvature of  $\mathbb{A}$  and we have  $\mathbb{F}_{(x,A)}(a, a') = 0$ ,  $\mathbb{F}_{(x,A)}(a, Y) = a(Y)$ ,  $\mathbb{F}_{(x,A)}(Y, Y') = F_A(Y, Y')$  for  $Y, Y' \in T_x M$ , and  $a, a' \in T_A \mathcal{A} \simeq \Omega^1(M, \text{ad}P)$ . The group  $\text{Aut}P$  acts on  $\mathbb{P}$  by automorphisms and  $\mathbb{A}$  is a  $\text{Aut}P$ -invariant connection. Thus the results of Sections 5.1 and 6 can be applied to this case. If  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$  then we have the  $\text{Aut}P$ -equivariant characteristic form  $p_{\mathcal{G}} \in \Omega_{\text{Aut}P}^{2r}(\mathcal{A})$ . If  $\mathcal{G}$  is a Lie group acting on  $P$  by automorphisms and also on a closed, oriented manifold  $C$  with  $\dim C = 2r - 2$ , and  $c: C \rightarrow M$  is a  $\mathcal{G}$ -equivariant map, then the form  $\varpi_c = \int_c p_{\mathcal{G}} \in \Omega_{\mathcal{G}}^2(\mathcal{A})$  can be written  $\varpi_c = \sigma_c + \mu_c$ , with  $\mu_c$  a co-moment map for  $\sigma_c$ . The explicit expression of this form is the following (see [15]). For  $A \in \mathcal{A}$ ,  $a, b \in T_A \mathcal{A} \simeq \Omega^1(M, \text{ad}P)$  and  $X \in \text{Lie}\mathcal{G}$  we have  $(\sigma_c)_A(a, b) = r(r-1) \int_c p(a, b, F, \binom{r-2}{\cdot \cdot \cdot}, F)$  and  $(\mu_c)_A(X) = -r \int_c p(v_A(X), F, \binom{r-1}{\cdot \cdot \cdot}, F)$ . In particular, if  $X \in \text{gau}P$  we have  $(\mu_c)_A(X) = -r \int_c p(X, F, \binom{r-1}{\cdot \cdot \cdot}, F)$ . The other forms of Section 6 can be computed in a similar way. A background connection is simply an element  $A_0 \in \mathcal{A}$ . The 1-form  $\rho_c = \int_c Tp(\mathbb{A}, \text{pr}_1^* A_0) \in \Omega^1(\mathcal{A})$  that determines the connection  $\Xi_c$  is given by  $(\rho_c)_A(a) = r(r-1) \int_c \int_0^1 p(A - A_0, ta, F_t, \binom{r-2}{\cdot \cdot \cdot}, F_t) dt$  with  $A_t = tA + (1-t)A_0$  and  $F_t$  the curvature of  $A_t$ . The form  $\beta_c = \int_c Tp(\mathbb{A}, \text{pr}_1^* A_0, \text{pr}_1^* A'_0)$  that appears in the change of background connection is simply given by  $\beta_c(A) = \int_c Tp(A, A_0, A'_0)$  (this follows from the tautological definition of  $\mathbb{A}$ ). Finally, if  $u: U \rightarrow M$  is a  $\mathcal{G}$ -invariant map such that  $c = \partial u$  then  $s_u = -\int_u Tp(\mathbb{A}, \text{pr}_1^* A_0)$  is given by  $s_u(A) = -\int_u Tp(A, A_0)$ . Also we have  $(\sigma_u)_A(a) = r \int_u p(a, F, \binom{2r-1}{\cdot \cdot \cdot}, F)$  for  $a \in T_A \mathcal{A} \simeq \Omega^1(M, \text{ad}P)$ .

**Remark 26** The connection  $\mathbb{A}$  and the expression of the differential forms can be obtained also from the (finite dimensional) bundle of connections (see [15]). This approach is specially useful in the case of Riemannian metrics, where the expressions are more complicated (see [17, 18]). Moreover, it is fundamental when dealing with the problem of locality in anomaly cancellation (see [16]).

## 7.1 Action by gauge transformations

Now we assume that a Lie group  $\mathcal{G}$  acts on  $P \rightarrow M$  by gauge transformations. In this case, as  $\mathcal{G}$  does not act on  $M$ , we can consider any smooth map  $c: C \rightarrow M$  with  $\dim C = 2r - 2$ . Furthermore,  $\int_c \Upsilon_{\mathbb{P}} = \Upsilon_{\mathbb{P}}/c$  is the cap product of  $[c] \in H^{2r-2}(M, \mathbb{Z})$  and  $\Upsilon_{\mathbb{P}} \in H^{2r}(M \times \mathcal{A}_{\mathcal{G}}, \mathbb{Z})$ . We summarize the results in the following

**Theorem 27** *Let  $P \rightarrow M$  principal  $G$ -bundle,  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ ,  $A_0$  a background connection on  $P$  and  $c: C \rightarrow M$  a smooth map with  $C$  closed and oriented and  $\dim C = 2r - 2$ . If  $\mathcal{G}$  acts on  $P$  by elements of  $\text{Gau}P$ , these data determine an action of  $\mathcal{G}$  on  $\mathcal{U}_c = \mathcal{A} \times U(1) \rightarrow \mathcal{A}$  by  $U(1)$ -bundle automorphisms such that the connection  $\Xi_c = \theta - 2\pi i \rho_c$  is  $\mathcal{G}$ -invariant, the equivariant curvature of  $\Xi_c$  is  $\varpi_c$  and  $c_{1,\mathcal{G}}(\mathcal{U}_c) = \Upsilon_{\mathbb{P}}/c \in H^2(\mathcal{A}_{\mathcal{G}}, \mathbb{Z})$ .*

*Furthermore, if  $c = \partial u$  and  $s_u(A) = -\int_u Tp(A, A_0)$ , then  $\alpha_{c,\phi}(x) = s_u(\phi x) - s_u(x)$ . Hence  $S_u = \exp(2\pi i \cdot s_u)$  determines  $\mathcal{G}$ -equivariant section of  $\mathcal{U}_c \rightarrow \mathcal{A}$  and we have  $\nabla^{\Xi_c} S_u = -2\pi i \sigma_u \cdot S_u$ .*

**Remark 28** In [22] the equation  $\alpha_{c,\phi}(A) = s_u(\phi A) - s_u(A) = -\int_u Tp(\phi A, A_0) + \int_u Tp(A, A_0)$  is used to define the action  $\alpha_{c,\phi}$ . To do this it is necessary to express the manifold  $c$  as the boundary of another manifold  $u$  and to extend the connections on  $c$  to  $u$ . In dimension two this can be done because cobordism is trivial, but this procedure cannot be generalized to higher dimensions.

## 7.2 Naturality of the prequantization bundle.

Let  $\mathcal{G}$  act on another principal  $G$ -bundle  $P' \rightarrow M'$  by  $G$ -bundle automorphisms, and let  $f: P \rightarrow P'$  be a  $\mathcal{G}$ -equivariant morphism of principal  $G$ -bundles, with projection  $\underline{f}: M \rightarrow M'$ . Then  $f$  induces a  $\mathcal{G}$ -equivariant map  $f_{\mathcal{A}}: \mathcal{A}' \rightarrow \mathcal{A}$ , and a map  $c' = \underline{f} \circ c: C \rightarrow M'$ . We want to prove that  $f_{\mathcal{A}}^*(\mathcal{U}_c) = \mathcal{U}_{c'}$ .

Choose a background connection  $A'_0$  on  $P'$ . Then  $A_0 = f_{\mathcal{A}}(A'_0)$  is a background connection on  $P$ . If  $\alpha'_{c'}$  and  $\Xi'_{c'}$  are the cocycle and connection determined by  $c'$  and  $A'_0$  we have  $\Xi'_{c'} = (f_{\mathcal{A}} \times \text{id}_{U(1)})^* \Xi_c$ . We show that the same is true for  $\alpha$ . Recall that the results of Section 6 are obtained choosing a connected and simply connected manifold  $E$  such that  $\mathcal{G}$  acts freely on  $\mathcal{A} \times E$ . If  $x_0 \in P$  and  $\text{Gau}P^{x_0}$  is the subgroup of gauge transformations that fix  $x_0$ , we have the exact sequence  $0 \rightarrow \text{Gau}P^{x_0} \rightarrow \text{Gau}P \rightarrow G \rightarrow 0$ , where the last map is evaluation on  $x_0$ . Hence it is enough to find a connected and simply connected manifold  $E$  in which  $G$  acts freely, for example  $EG$  or a finite dimensional approximation (for example, if  $G$  is connected and simply connected we can simply take  $E = G$ ). Choose  $x_0 \in P$  and set  $x'_0 = f(x_0)$ . The points  $x'_0$  and  $x_0$  determine free actions of  $\mathcal{G}$  on  $\mathcal{A}' \times E$  and  $\mathcal{A} \times E$  and  $f_{\mathcal{A}} \times \text{id}_E$  is  $\mathcal{G}$ -equivariant. If  $\mathfrak{A}$  is a connection on  $\mathcal{A} \times E \rightarrow (\mathcal{A} \times E)/\mathcal{G}$  then  $\mathfrak{A}' = (f_{\mathcal{A}} \times \text{id}_E)^* \mathfrak{A}$  is a connection on  $\mathcal{A}' \times E \rightarrow (\mathcal{A}' \times E)/\mathcal{G}$  and using the naturality of integrated Chern-Simons characters of Section 5.5 it can be proved that  $\alpha'_{c',\phi} = f_{\mathcal{A}}^*(\alpha_{c,\phi})$  for any  $\phi \in \mathcal{G}$ .

## 7.3 Flat connections

If  $\mathcal{F} \subset \mathcal{A}$  is the subspace of flat connections, for  $r \geq 2$  we have  $\mathcal{F} \subset \mu_c^{-1}(0)$ . In particular the restriction to  $\mathcal{F} \times U(1)$  of the form  $\Xi_c$  is  $\mathcal{G}$ -basic. If  $\mathcal{G}$  acts freely on  $\mathcal{F}$  and we have a quotient bundle  $(\mathcal{F} \times U(1))/\mathcal{G} \rightarrow \mathcal{F}/\mathcal{G}$  then  $\Xi_c$  projects onto a connection  $\Xi_c$  on this bundle, and the curvature of  $\Xi_c$  is the pre-symplectic

form  $\underline{\sigma}_c$  on  $\mathcal{F}/\mathcal{G}$  obtained by symplectic reduction to  $\mathcal{F}$  from  $(\mathcal{A}, \varpi_c)$ . Hence we obtain a prequantization for  $(\mathcal{F}/\mathcal{G}, \underline{\sigma}_c)$ . For  $r = 2$  and  $C = M$  a closed oriented surface, we obtain  $(\sigma_M)_A(a, b) = 2 \int_M p(a, b)$ ,  $(\mu_M)_A(X) = -2 \int_M p(X, F)$  and  $(\rho_M)_A(a) = \int_M p(A - A_0, a)$ , for  $A \in \mathcal{A}$ ,  $a, b \in T_A \mathcal{A} \simeq \Omega^1(M, \text{ad} P)$  and  $X \in \text{Lie} \mathcal{G}$ . If  $p: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  is a non-degenerate bilinear form, then  $\sigma_M$  is a symplectic form and the moment map can be identified with the curvature map  $A \mapsto F$ . Hence they coincide with the symplectic structure and moment map defined on [3]. As commented in Remark 28 in this case our bundle also coincides with that of [22], and the connection  $\Xi_M$  projects onto a connection on the quotient bundle  $(\mathcal{F} \times U(1))/\mathcal{G} \rightarrow \mathcal{F}/\mathcal{G}$ . If  $J$  is a complex structure on  $M$ , it induces a complex structure on  $\mathcal{A}$  and  $\sigma_M$  is of type  $(1, 1)$ . As  $\nabla^{\Xi_M}$  is a unitary connection we conclude (see [13]) that it determines a holomorphic structure on  $\underline{\mathcal{L}}_M \rightarrow \mathcal{F}/\mathcal{G}$ . We have similar results when  $\dim M > 2$  and  $c: C \rightarrow M$  is a map with  $\dim C = 2$ . If  $c = \partial u$ , the restriction of  $S_u$  to  $\mathcal{F}$  is a  $\Xi_c$ -parallel section as it satisfies  $\nabla^{\Xi_c} S_u = 0$  because  $(\sigma_u)_A(a) = 2 \int_u p(a, F) = 0$  if  $A \in \mathcal{F}$ .

If  $r \geq 3$  we have  $\sigma_c|_{\mathcal{F}} = 0$ , and in this case  $\Xi_c$  is a flat connection, and hence its holonomy defines a cohomology class in  $H^1(\mathcal{F}/\mathcal{G}, \mathbb{R}/\mathbb{Z})$  (see [10] for a generalization of this result to arbitrary dimensions).

## 7.4 Irreducible connections

We denote by  $\tilde{\mathcal{A}}$  the space of irreducible connections. Although  $\text{Gau} P$  does not act freely on  $\tilde{\mathcal{A}}$ , the isotropy group is the same  $Z(G)$  (the center of  $G$ ) for all  $A \in \tilde{\mathcal{A}}$ , and  $\tilde{\mathcal{A}}/\text{Gau} P$  is a differential manifold. If we define the group  $\tilde{\mathcal{G}} = \text{Gau} P/Z(G)$  then  $\tilde{\mathcal{G}}$  acts freely on  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}/\tilde{\mathcal{G}}$  is a principal  $\tilde{\mathcal{G}}$ -bundle (see for example [13] for details). In the preceding section we have constructed a  $\text{Gau} P$ -equivariant prequantization bundle  $\mathcal{U}_c \rightarrow \mathcal{A}$ . If we restrict it to  $\tilde{\mathcal{U}}_c = \tilde{\mathcal{A}} \times U(1) \rightarrow \tilde{\mathcal{A}}$  we hope that it will define a prequantization bundle over  $\tilde{\mathcal{A}}/\tilde{\mathcal{G}}$ , but there is a problem: the action of  $Z(G)$  on  $\tilde{\mathcal{U}}_c$  does not need to be trivial and  $\tilde{\mathcal{G}}$  does not act on  $\tilde{\mathcal{U}}_c$ . Or, in an equivalent way,  $\tilde{\mathcal{U}}_c/\text{Gau} P \rightarrow \tilde{\mathcal{A}}/\text{Gau} P$  is not a  $U(1)$ -bundle. If the action of  $Z(G)$  on  $\mathcal{U}_c$  is trivial then  $\tilde{\mathcal{G}}$  acts on  $\mathcal{U}_c$ , and restricting  $\tilde{\mathcal{A}}$  to  $\tilde{\mathcal{U}}_c/\tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{A}}/\tilde{\mathcal{G}}$  we obtain a bundle over the moduli space of irreducible connections. This is the case for the trivial  $SU(2)$ -bundle over a surface, as it is shown on [22]. If the action of  $Z(G)$  on  $\mathcal{U}_c$  is not trivial, we can define  $\tilde{G} = G/Z(G)$  and  $\tilde{P} = P/Z(G) \rightarrow M$ , which is a principal  $\tilde{G}$ -bundle. We also set  $\tilde{\mathbb{P}} = (P/Z(G)) \times \tilde{\mathcal{A}}$  which is also a principal  $\tilde{G}$ -bundle. The connection  $\mathbb{A} \in \Omega^1(\mathbb{P}, \mathfrak{g})$  induces a connection  $\tilde{\mathbb{A}}$  on  $\tilde{\mathbb{P}}$  which is invariant under the action of  $\tilde{\mathcal{G}}$  (see [10] for details). The results of Section 6 can be applied to the bundle  $\tilde{P} = P/Z(G) \rightarrow M$ ,  $N = \tilde{\mathcal{A}}$  and the  $\tilde{\mathcal{G}}$ -invariant connection  $\tilde{\mathbb{A}}$  on  $\tilde{\mathbb{P}}$ , and we obtain a result analogous to Theorem 27, but we should take polynomials and characteristic classes of  $\tilde{G}$  in place of  $G$ . If  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(\tilde{G})$  and  $c: C \rightarrow M$  with  $\dim C = 2r - 2$ , we obtain  $\varpi_c \in \Omega_{\tilde{\mathcal{G}}}^2(\tilde{\mathcal{A}})$  and a  $\tilde{\mathcal{G}}$ -equivariant prequantization bundle  $(\tilde{\Xi}_c \tilde{\mathcal{U}}_c)$  of  $(\tilde{\mathcal{A}}, \varpi_c)$ , and taking the quotient a  $U(1)$ -bundle  $\tilde{\mathcal{U}}_c/\tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{A}}/\tilde{\mathcal{G}}$ . If  $\tilde{\mathcal{F}} \subset \mu^{-1}(0)$  is the space of irreducible flat connections,  $\tilde{\Xi}_c$  projects onto a

connection on this bundle and we obtain a prequantization bundle of  $(\tilde{\mathcal{F}}/\tilde{\mathcal{G}}, \underline{\varrho}_c)$ , where  $\underline{\varrho}_c$  is obtained from  $\varpi_c$  by symplectic reduction.

We consider only one example. If  $G = SU(2)$  then  $\tilde{G} = SO(3)$ . Both groups have the same Lie algebra  $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ . As they are connected, they have the same Weil polynomials  $I(SU(2)) = I(SO(3))$ , but  $I_{\mathbb{Z}}(SO(3)) \subsetneq I_{\mathbb{Z}}(SU(2))$ . For example the second Chern polynomial  $c_2 \notin I_{\mathbb{Z}}(SO(3))$ , but the first Pontryagin polynomial  $p_1 = 4c_2 \in I_{\mathbb{Z}}(SO(3))$  (see [12, Formula 4.11]). If  $c: C \rightarrow M$  is a map with  $C$  a closed surface, the pre-symplectic structure  $\underline{\varrho}_c$  on the moduli space of irreducible flat connections determined by  $c$  and the second Chern class may not be prequantizable. But  $4 \cdot \underline{\varrho}_c$  is always prequantizable by the bundle associated to the first Pontryagin class.

## 7.5 The space of connections and automorphisms

Let  $\text{Aut}^+ P$  be the group of automorphisms preserving the orientation on  $M$ , and assume that  $\mathcal{G}$  is a group acting on  $P$  by elements of  $\text{Aut}^+ P$ . The only difference with the case of gauge transformations is that we cannot choose  $c: C \rightarrow M$  an arbitrary map because it should be  $\mathcal{G}$  invariant. We only consider the cases  $C = M$  (if  $\partial M = 0$ ) and  $C = \partial M$ .

### 7.5.1 Base manifold closed

When  $M$  is a closed manifold of dimension  $2r - 2$ , we can take  $C = M$  and  $c = \text{id}_M$ , which clearly is  $\mathcal{G}$ -invariant. As a consequence of Theorem 17 we obtain the following

**Theorem 29** *Let  $P \rightarrow M$  principal  $G$ -bundle with  $M$  closed oriented and  $\dim M = 2r - 2$ ,  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ ,  $A_0$  a background connection on  $P$  and a group  $\mathcal{G}$  acting on  $P$  by elements of  $\text{Aut}^+ P$ . These data determine an action of  $\mathcal{G}$  on  $\mathcal{U}_M = \mathcal{A} \times U(1) \rightarrow \mathcal{A}$  by  $U(1)$ -bundle automorphisms such that the connection  $\Xi_M = \theta - 2\pi i \rho_M$  is  $\mathcal{G}$ -invariant, the equivariant curvature of  $\Xi_M$  is  $\varpi_M$ , and  $c_{1,\mathcal{G}}(\mathcal{U}_M) = \int_M \Upsilon_{\mathbb{P}} \in H^2(\mathcal{A}_{\mathcal{G}}, \mathbb{Z})$ .*

Theorem 29 extends to arbitrary bundles the results of [1, 2] for trivial bundles over a surface.

### 7.5.2 Base manifold with boundary

Now we assume that  $M$  is a compact oriented manifold of dimension  $2r - 1$  with boundary  $\partial M$ . We chose  $C = \partial M$  and  $c = \text{id}_M$ . By applying Theorem 17 we obtain the following

**Theorem 30** *Let  $P \rightarrow M$  principal  $G$ -bundle with  $M$  compact and oriented with boundary  $\partial M$  and  $\dim M = 2r - 1$ . If a group  $\mathcal{G}$  acts on  $P$  by elements of  $\text{Aut}^+ P$ ,  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$  and  $A_0$  is a background connection on  $P$ , these data determine an action of  $\mathcal{G}$  on  $\mathcal{U}_{\partial M} = \mathcal{A} \times U(1) \rightarrow \mathcal{A}$  by  $U(1)$ -bundle automorphisms such that the connection  $\Xi_{\partial M} = \theta - 2\pi i \rho_{\partial M}$  is  $\mathcal{G}$ -invariant, the equivariant curvature of  $\Xi_{\partial M}$  is  $\varpi_{\partial M}$ , and  $c_{1,\mathcal{G}}(\mathcal{U}_{\partial M}) = \int_{\partial M} \Upsilon_{\mathbb{P}} \in H^2(\mathcal{A}_{\mathcal{G}}, \mathbb{Z})$ .*

Furthermore,  $S_M = \exp(-2\pi i \cdot \int_M Tp(A, A_0))$  determines  $\mathcal{G}$ -equivariant section of  $\mathcal{U}_{\partial M} \rightarrow \mathcal{A}$  and we have  $\nabla^{\Xi_{\partial M}} S_M = -2\pi i \sigma_M \cdot S_M$ .

## 8 First order integrated Chern-Simons characters

### 8.1 Dijkgraaf-Witten action for Chern-Simons theory

Usually classical Chern-Simons theory is defined for trivial bundles over 3-manifolds using global sections, but this procedure can not be generalized to non-trivial bundles. In [12] Dijkgraaf and Witten shown how the Chern-Simons action can be defined in this case using differential characters.

Let  $A$  be a connection on  $P \rightarrow M$ ,  $(p, \Upsilon) \in \mathcal{I}_{\mathbb{Z}}^r(G)$ , and let  $\chi^A \in \hat{H}^{2r}(M)$  be the Chern-Simons character of  $A$  associated to  $(p, \Upsilon)$ . If  $c: C \rightarrow M$  is a smooth map with  $C$  closed and  $\dim C = 2r - 1$  we define the Chern-Simons action  $\lambda_c: \mathcal{A} \rightarrow \mathbb{R}/\mathbb{Z}$  by  $\lambda_c(A) = \chi_A(c)$ . By definition, if  $c = \partial u$  we have  $\lambda_{\partial u}(A) = \chi_A(\partial u) = \int_u p(F, \dots, F)$ . If  $A_0$  is another connection on  $P$  we have  $\lambda_c(A) - \lambda_c(A_0) = \chi^A(c) - \chi^{A_0}(c) = \int_c Tp(A, A_0)$  and hence  $\lambda_c(A) = \lambda_c(A_0) + \int_c Tp(A, A_0)$ . Note that  $\lambda_c$  gives a definition of the Chern-Simons action which is independent of the background connection  $A_0$  chosen, i.e., the characteristic class  $\Upsilon$  fixes the value of  $\lambda_c(A_0)$ . As remarked in [12] this constant is relevant in the quantization of the theory. It can be seen that  $\lambda_c$  is  $\text{Gau}P$ -invariant. Furthermore, if  $M$  is closed and  $\dim M = 2r - 1$  then  $\lambda_M$  is  $\text{Aut}^+P$ -invariant, and the same happens for  $\lambda_{\partial M}$  if  $M$  has boundary and  $\dim M = 2r$ .

### 8.2 First order integrated Chern-Simons characters

Now we show how first order integrated Chern-Simons Lagrangian are related to  $\lambda_c$ . First we consider the free action case. We show in Section 5.3 that  $\int_c \chi_{\mathfrak{A}} \in \hat{H}^1(\mathcal{A}/\mathcal{G})$  does not depend on  $\mathfrak{A}$ . In accordance with Section 4.1  $\int_c \chi_{\mathfrak{A}}$  determines a function  $\varphi_c: \mathcal{A} \rightarrow \mathbb{R}/\mathbb{Z}$  given by  $\varphi_c(A) = \int_c \chi_{\mathfrak{A}}([A]) = \chi_{\mathfrak{A}}(c \times [A])$ .

The connection  $A$  induces a morphism of principal  $G$ -bundles  $g_A: P \rightarrow (P \times \mathcal{A})/\mathcal{G}$ , defined by  $g_A(y) = [(y, A)]$  for  $y \in P$ . By Proposition 4 we have  $g_A^*(\Lambda_{\mathfrak{A}}) = A$ , and hence  $\underline{g}_A^* \chi_{\mathfrak{A}} = \chi^A$ , where  $\underline{g}_A: M \rightarrow M \times (\mathcal{A}/\mathcal{G})$  is given by  $\underline{g}_A(x) = (x, [A])$ . We conclude that  $\lambda_c(A) = \chi^A(c) = \underline{g}_A^* \chi_{\mathfrak{A}}(c) = \chi_{\mathfrak{A}}(c \times [A]) = \varphi_c(A)$ .

We can use the same arguments than the used for second order characters to show that  $\varphi_c$  is also well defined for non-free actions, and that it coincides with  $\lambda_c$ .

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